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THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

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PUBLISHED BY
THE AMERICAN PHYSICAL SOCIETY
LANCASTER, PA., AND ITHACA, N. Y.

Entered at the post-office, at Lancaster, Pa., as second class matter

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THE PHYSICAL REVIEW, ITHACA, N. Y.

Second Series *May, 1920* *Vol. XV., No. 5*

THE

PHYSICAL REVIEW.

MAGNETIZATION AND HYSTERESIS IN HEMATITE CRYSTALS.

By T. Townsend Smith.

Synopsis.

Magnetization and Hysteresis in Hematite Crystals; Experimental Procedure.—In a previous paper, the author has given the results of an investigation of the magnetic properties of crystallized hematite, the measurements being of the force exerted upon a sphere when it is placed in a non-uniform magnetic field. The present paper contains some results obtained by measuring the couple exerted when a hematite sphere is suspended in a uniform magnetic field. The couple in a uniform field measures the component of magnetization perpendicular to the field; the force measures the component parallel to the field.

Experimental Data.—The paper contains (1) *magnetization curves* (Fig. 4) for two specimens of hematite; (2) *hysteresis loops* at fields from 9.5 to 1020 gausses, some obtained with a rotating field (Figs. 6, 7, 8, 9, 11), the others with an alternating field (Figs. 10, 11); (3) a graph representing *components of magnetization* parallel and perpendicular to the field (Fig. 12).

Deduction, The normal magnetization curve of hematite consists of a sharp rise followed by a slow increase, which is proportional to the field strength. The *hysteresis energy loss* increases with an increase in the maximum field applied up to a certain field, and thereafter the area of the hysteresis loop remains constant. For the specimen on which most of the present measurements were made, the hysteresis loss is constant for field strengths above about 150 gausses.

In a previous paper[1] the author has published the results of an investigation on the magnetic properties of hematite crystals. In that work it was shown that certain hematite crystals possessed a dual magnetic character, exhibiting hysteresis in the principal plane of the crystal, but being apparently free from hysteresis in a direction at right angles to this plane. It was further shown that the intensity does not even approach saturation at the highest field investigated, just under 4,000 gausses.

The crystals investigated consisted of a number from the Isle of Elba,

[1] Phys. Rev. (2), 8: 721, 1916.

and specimens from Ouropreto, Brazil, from Schabry in the Ural Mountains, and from Dognacska, Hungary (a twin).

The results presented in the previous paper were all based upon measurements of the pull exerted upon the spheres in a non-uniform magnetic field, and the magnetic quantity obtained was the component of the intensity parallel to the field. In the present paper some results are given which were obtained by measuring the couple exerted upon the spheres when they were suspended in a uniform field, which could be rotated at will around the axis of suspension. From this couple is obtained the component of magnetic intensity perpendicular to the exciting field.

The results of the present work verify the previous work with reference to the paramagnetic behavior along the axis for three of the specimens previously reported, but the increased sensitiveness of the torsion measurements has demonstrated that the twin crystal from Dognacska and all of the Elba crystals exhibit at least a slight amount of hysteresis along the axis.

In the case of the Brazil and Ural Mountain specimens further interesting properties have been discovered. These are: (1) the specimens behave as if the magnetization curve were represented by the equation

$$I = \pm (I_0 + kH), \tag{1}$$

inasmuch as a comparatively large value is quickly reached and the intensity then increases slowly in proportion to the field strength. This is the combination of permanent magnetism and paramagnetism which is sometimes used as a mathematical supposition.

There is (2) the further property of a maximum area to hysteresis in the principal plane. This means that both the energy loss and the coercive force reach maxima at comparatively low fields.

1. *Couple Exerted on a Suspended Crystal.*—If a magnetically non-isotropic sphere be suspended in a uniform field, the direction of magnetization will coincide with that of the field only when the direction of maximum susceptibility or of minimum susceptibility coincides with the direction of the field. For all intermediate positions the direction of the magnetization will make an angle with the exciting field, the direction of magnetization being closer than that of the field to the direction of maximum susceptibility. The sense of the component of magnetization perpendicular to the field will, therefore, be such as will tend to rotate the crystal in that direction which will make the direction of maximum susceptibility coincide with that of the exciting field. There will consequently be exerted upon the specimen a couple, which will act against

the torsional moment of the suspending wire in an effort to set the direction of maximum susceptibility parallel to the field.

Such at least is the simple phenomenon which is exhibited by crystals in which hysteresis is lacking, but in general with hysteresis present the position of no torque might be oblique to the direction of maximum permeability, on account of the component of the residual magnetization which is perpendicular to the field. There is one special case where this will not be true, and fortunately that is the one which is here dealt with—the case in which there is a direction of minimum susceptibility along which there is no hysteresis.

It does not seem to me to follow necessarily from this that the component of the exciting field in the direction of the principal plane will have the same effect as if the other component did not exist. The experimental results, however, seem to indicate that this is at least approximately true.

In Fig. 1 the hematite sphere is supposed to be suspended about an

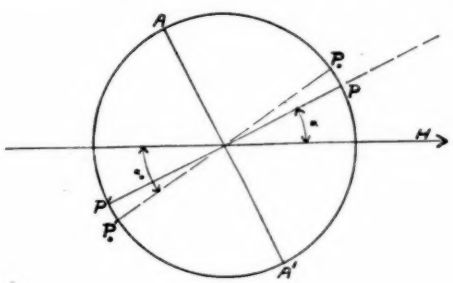

Fig. 1.

axis, which lies in the principal plane of the crystal, and which is, therefore, perpendicular to the plane of the sketch. PP' is the trace of the magnetic principal plane, and AA' is the direction of the magnetic axis of symmetry. The angle between H and PP' is α, and this angle will, of course, tend to diminish as the field increases. The value of α for a zero field strength is indicated by α_0, so that when the field is applied the sphere rotates through the angle $(\alpha_0 - \alpha)$.

If the torsion constant of the suspending wire be τ, then the couple exerted by the spring is $\tau(\alpha_0 - \alpha)$ dynes $\times$ cm. and the couple exerted by the field on a sphere, the volume of which is "V" c.c., is equal to $H \cdot I \cdot \sin(H, I) \cdot V$, or more simply, for a unit volume, the product of the field intensity into that component of the intensity of magnetization which is perpendicular to the field. This component can be readily written down and is $I_P \sin \alpha - I_A \cos \alpha$ which gives as the condition

for equilibrium

$$\tau(\alpha_0 - \alpha) = H[I_P \sin \alpha - I_A \cos \alpha]v. \tag{2}$$

It is known that the material here investigated is paramagnetic in the direction of AA' and the susceptibility in this direction has been determined to be $k = .000117$ (loc. cit., page 731), so that the component of the intensity in this direction may be calculated and the equation (1) may then be solved for I_P. This is the procedure which has been followed.

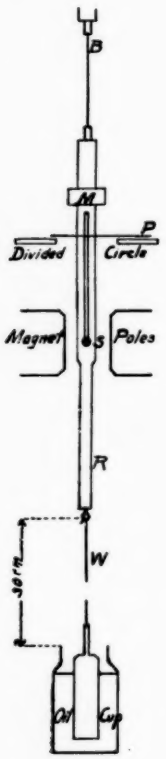

Fig. 2.

Suspension of Specimen.

B, Spring brass wire; *M*, Small mirror for telescope and scale; *P*, Glass capillary used for alidade; *S*, Sphere of hematite; *R*, Rod of maple, half cut away and grooved along its axis; *W*, Light brass rod, used to support the steadying and damping weight, which dipped into the oil cup beneath the table.

2. *The Suspension of the Crystal.* — The arrangement of the apparatus for suspending a sphere in a magnetic field is shown in Fig. 2. In the previous work the sphere was attached by means of a non-magnetic cement called cæmentium to a light wooden rod, which was about 3 or 4 millimeters in diameter, was about 7 or 8 centimeters long, and carried a light pointer of glass. In the present work the spheres are mounted on these wooden rods as before, and the rods are clipped with a little spring of brush copper to a heavier rod of maple. This maple rod was supported by a suspending wire of spring brass and was held plumb by a brass weight below. This weight dipped into a wide-mouthed bottle, which contained machine oil, and hence the weight served the additional purpose of damping the vibrations. The maple rod (R) could be unhooked at W and thus be lifted bodily out of the apparatus for the purpose of changing the specimen or making any other necessary adjustments.

The rod (R) was first turned to the desired diameter, half of it cut away along its whole length, with the exception of short distances for suspending purposes at the two ends, and then grooved along the axis. The rod was nicked opposite P to make a rest for the glass pointer, and a hollow was scooped out at S to make room for the spheres. The sphere to be investigated was thus situated in the axis of rotation of the suspending system, and the position of the sphere with reference to the field could be read by the pointer P on the divided circle, which was rigidly fastened

to the electromagnet and rotated with the field. The magnet was mounted on a turn table, and was rotated by a bicycle-chain drive from outside of the box containing the apparatus.

The little mirror M was fastened to the rod R with beeswax and was used with a telescope and scale for measuring the deflections produced when the magnetic field was applied. In some of the work a second small mirror was placed just alongside of M, but attached to the divided circle, so that small angles of rotation of the magnet could be read accurately, using the same telescope and scale as was used for the deflections.

The suspension was altogether of non-magnetic materials, wood, brass, and copper, and gave a barely appreciable deflection with the sphere removed even at the strongest fields which were used. When first cut out the rod R was shellacked, but the shellac was sufficiently magnetic to give trouble so that it had to be carefully sandpapered off. The maximum deflection was then not greater than one or two tenths of a millimeter for a field of about 1,000 gausses, using a suspension whose torsion constant was 1.90 dynes $\times$ centimeters per centimeter deflection. The error here can be safely neglected, being in no case greater than about one part in 300.

3. *Calibrations.*—In order to make a determination of the magnetic intensity in the principal plane it was, according to equation (2), necessary to determine four quantities experimentally. These were

the torsion constant τ,
the angular deflection $(\alpha_0 - \alpha)$,
the field strength H,
the angle between principal plane and field, α.

The *torsion constants* were determined in two ways—first by means of a torsional pendulum, and secondly, by means of the torque exerted upon a small, flat coil of wire, which was suspended in the field and through which a small electric current was passed. This coil was mounted on a rod similar to R and this rod could be put in place of R when desired. The coil was designed for use in calibrating the magnetic field of the magnet, but after this work had been done and the magnetic field strength was known, the known field strength and the calibrated coil could be used for determining the torsion constant.

Two or three suspensions were broken during the course of the work, and it is unfortunately true that the torsion constant in several cases was not known with the accuracy which was thought to be the case at the time.

The *magnetic field strength* determinations in almost every case were

made with the flat coil just referred to. Current leads for this coil were provided by the spring brass wire *B* and a fine coil of rolled galvanometer suspension, which was clipped to *W*. A current of about one thirtieth of an ampere was passed through this coil which was placed with its plane parallel to the direction of the magnetic field. The deflections produced were thus proportional to the field strength. The constant of proportionality was determined by means of an exploring coil and a standard of mutual inductance, both of which were used in the previous work.

In this way the magnetic field was determined under a variety of conditions. The magnetization curve for the electromagnet was determined, starting with the magnet carefully demagnetized by reversals, and then hysteresis loops for maximum fields of about 92, 190, 238, 335, 450, and 620 gausses were obtained in order that the specimens could be taken through such ranges with alternating fields.

I. Magnetization Values.

4. *The Magnetization Curves—Direction of the Axis.*—The torsional measurements afford a very delicate test of the paramagnetic character of the spheres in the direction of the axis, because a very small component of the magnetization in this direction can be detected by the couple which it exerts when the specimen is placed with the principal plane parallel to the field. If there is no hysteresis along the axis, then it will be possible to find the principal plane by setting for a position of no deflection or no torque. This will be the case when the axis of the crystal is perpendicular to the field, in which position (axial hysteresis assumed absent) there will be no rotation of the specimen caused by the field, even when the field is reversed in direction. If there is hysteresis along the axis, it will always be possible to find a direction for the field for which there will be no rotation when the field is applied, but in this case a reversal of the field causes a rotation of the specimen and it will be impossible to find a position for which the sphere will remain in its zero position when the direction of the field is thus changed.

The difference between the two cases may be seen by an inspection of Fig. 3, in which deflections in centimeters (proportional to the torque) are plotted against angular readings on the divided circle (the angle α of Fig. 1). The points for sphere *G*, axially paramagnetic, are for field strengths of 76 and 1,020 gausses, as indicated on the figure. The large circles represent a rotation with increasing positive angles, the smaller circles an opposite rotation. Concentric circles mean that the readings taken with opposite rotations are identical. At 1,020 gausses

the barred circles represent angles in the neighborhood of 180° and the open circles points near 0°. These two sets do not quite coincide, though in each set the reversal of the field obviously does not change the deflection. No precise setting of the specimen at the center of the divided circle was attempted, and the discrepancy between the two sets is doubtless due to the axis of suspension being slightly to one side of the axis of rotation of the magnet. The points on the two curves

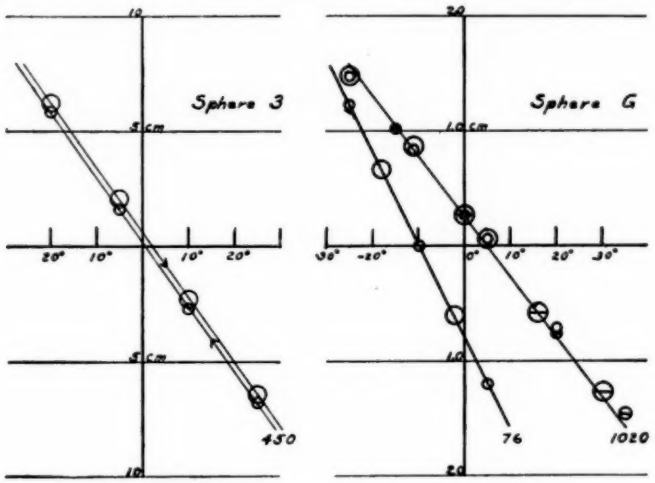

Fig. 3.

Axial Hysteresis.

At left —dependence of torque upon direction of rotation for sphere 3, Elba hematite.
At right—lack of axial hysteresis in sphere *G*.

for field strengths of 1,020 and 76 were taken with different set-ups, which accounts for the fact that the angle for no torque is different in the two cases. The readings for sphere 3, which were taken with the express purpose of investigating this question, show clearly the presence of hysteresis.

The net result of such tests is that there are three of the spheres which have shown themselves to be without hysteresis along the axis—*G* from Brazil, and J_1 and J_2 from the Ural Mountains. Hysteresis along the axis is exhibited by all specimens[1] of Elba Hematite so far tested, sphere 3 being an Elba specimen for which the hysteresis is slight. Sphere *F*, cut from a twin crystal from Dognacska, showed a slight trace of axial hysteresis.

5. *Magnetization Curves—Plane of Symmetry.*—Values for the intensity of magnetization for spheres *G* and *J* are shown in Fig. 4, all of the determinations being obtained from the torsion exerted in a uniform

[1] Loc. cit., p. 733.

field. The magnetic plane was set at a convenient, small angle (α) with the field, and the values of I_p were then determined with the aid of equation (2) above. The results are in fair agreement with the previous determinations made with a non-uniform field. (Loc. cit.)

The values for sphere G were determined in several runs with different set-ups:

(a) Field strengths up to about 100 gausses were obtained with two short solenoids, without the use of iron. These coils were mounted on

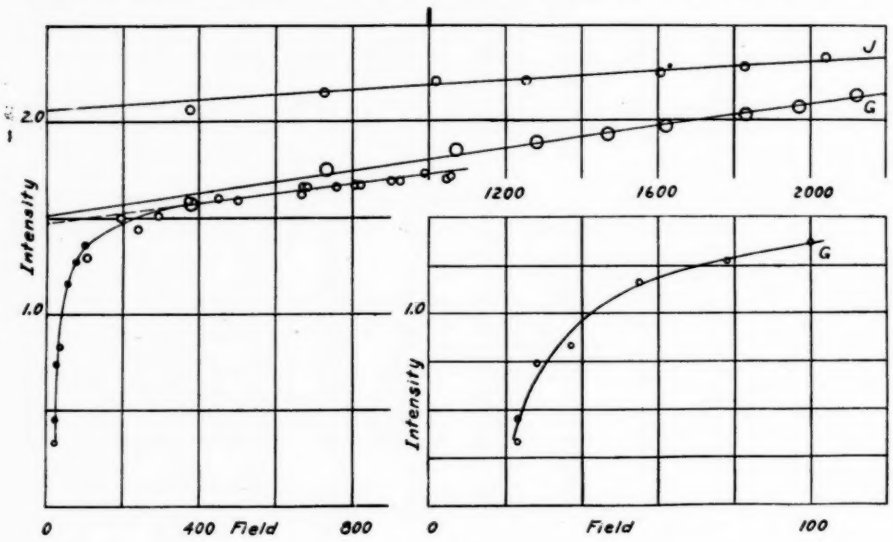

Fig. 4.

Magnetization Curves.

Lower curve for G—angle α small. The low field values, represented by the smallest circles are shown to a larger scale in the lower right-hand corner. These points are corrected for the demagnetizing field ($4\pi I/3$). The other points are not so corrected.

Upper curve for G—angle α is approximately 45°.

Top curve, J—angle α is approximately 45°.

the turntable in place of the electromagnet, coaxially and with just enough space between the two to make room for the rod R of Fig. 2. The points so determined are marked with the smallest circles in Fig. 4. The torsion constant of the suspending wire, number 35 brass about 9 centimeters long, was about 1.24 dynes × centimeters per centimeter deflection (scale distance about 72.5 centimeters) and the value of α_0 was about 8 degrees, the deflection being about half a degree for the largest field values.

(b) For the fields up to about a thousand gausses, the electro-magnet used in the previous work was fitted with a pair of flat-faced polepieces. The torsion constant was a little greater, 1.90 dynes × centimeters,

and the value of α_0 was varied from four to eight degrees without making any appreciable difference in the results obtained.

(*c*) To obtain fields above 1,000 gausses a second pair of pole pieces with a smaller air gap was used and the values shown by the large circles for sphere *G* and the values shown above for sphere *J* were so obtained. In this case, to eliminate the inaccuracy in the small initial values of α_0, the principal plane was set at an angle of 45 degrees to the field, readings being taken in all four quadrants to avoid chance errors in the reading of the angles. The maximum field strength reached was about 3,000, and the torsion constant of the suspending wire[1] was about 63.0 dynes $\times$ centimeters.

The interpretation of these measurements rests, of course, on the assumption that the components of the field along and perpendicular to the axis of symmetry of the crystal may be treated independently. The concordance between these results and the results of the author's previous work[2] makes this assumption very probable.

Making this assumption, then, the values of the intensity of magnetization were calculated for various field strengths as is shown in Table I.

TABLE I.

Magnetization Values for Sphere G.

Initial Angle (α_0) 45°.

I. Applied Field, H.	II. Deflection, Cm.	III. Reduced Torque, Dyne × Cm.	IV. Angle, α.	V. Apparent Intensity, $T/H \sin \alpha$.	VI. H_P, $H \cos \alpha$.	VII. I_P.
530....	0.98 (1)	570	44° 37′	1.53	377	1.57
1,020....	2.03 (3)	1,180	44° 12′	1.66	733	1.75
1,480....	3.06 (1)	1,775	43° 48′	1.73	1,070	1.85
1,760....	3.64 (4)	2,110	43° 32′	1.74	1,278	1.89
2,020....	4.21 (1)	2,445	43° 20′	1.76	1,468	1.93
2,220....	4.65 (4)	2,700	43° 10′	1.78	1,620	1.97
2,500....	5.33 (4)	3,100	42° 53′	1.82	1,830	2.03
2,680....	5.79 (2)	3,350	42° 43′	1.84	1,970	2.07
2,880....	6.31 (4)	3,670	42° 30′	1.88	2,120	2.13

The table is self explanatory, save perhaps for columns III. and VII. The values in column III. are obtained from those in column II. by multiplying by the torsion constant 63.0, and dividing by the volume of the sphere, 0.1092 cubic centimeters. The numbers in column VII. are obtained by adding $.000117^2 \times H_P$ to the numbers of column V. to

[1] Number 26 brass about 9 centimeters long.

[2] Loc. cit.

[3] Loc. cit., p. 731.

allow for the paramagnetism along the axis of the crystal. The numbers in parentheses in column II. are the number of readings on which the deflection is based. A least squares calculation gives

TABLE II.

Magnetization Values for Sphere J_1.

Initial Angle (α_0) 45°.

Field (H_P).	376.	724.	1,018.	1,252.	1,605.	1,827.	2,040.	Mean.
Apparent Intensity	2.01	2.06	2.09	2.06	2.06	2.07	2.09	2.07
Intensity (I_P)	2.06	2.15	2.21	2.21	2.25	2.28	2.33	

$$I = 1.518 + .000291 \cdot \times H \tag{3}$$

as the magnetization curve, assuming the form (1), with a probable error of .001 in I_0 and of .0000002 in K.

An attempt was made to fit the data to the form

$$I = I_0 + K_1H + K_3H^3 \tag{4}$$

for the term in H^2 is obviously lacking, but the value of K_3 so obtained was so small that at a field of 3,000 the deflection due to this factor would be .003 centimeters, which is beyond the range of the experimental accuracy. It may be regarded as established, then, that so far as these observations go the form of equation (1) fits the facts.

This is even more strikingly shown in the J specimen, Table 2, where the apparent intensity of magnetization remains constant within the range from 700 to 2,000 gausses, and which exhibits, therefore, the curious phenomenon of a substance having a uniform paramagnetism superposed upon a ferromagnetism, the ferromagnetism conforming to the crystal structure.[1]

The rather striking agreement between (1) and the observed values induced me to collect all the data on hematite which bears upon this question, and this data is shown in Table III. The "force" values (column V.) are taken from the author's previous paper on hematite and Fig. 4 shows three out of the four "torsion" values.

This coexistence of ferromagnetism and paramagnetism in the plane of symmetry in hematite has been previously observed in other materials. Frivold[2] gives some susceptibility values for silver, which, when reduced

[1] The statement made in the earlier article that the magnetization values for spheres 3, 5, J_1, and J_2 would lie in good agreement with those for G is not contradictory to the present work, for the values referred to were in the range from 2,000 to 4,000 gausses, in which range the two spheres (G and J_1) have nearly the same values of intensity.

[2] Ann. d. Physik, 57: 471, 1918.

TABLE III.

Magnetization Values for Hematite.

Plane of Symmetry of Crystal.

I.	II.	III.	IV.	V.
Specimen.	Values of Constants.		Range of Field Strength Gausses.	Method of Experimenting.
	I_0.	k.		
G.....	1.48	.00025	500–4,000	Force
G.....	1.48	.00025	400–1,100	Torsion
G.....	1.52	.00029	500–2,100	Torsion
J.....	1.97(?)	.00012	2,000–4,000*	Force
J.....	2.05	.00012	350– 620	Torsion
J.....	2.07	.00012	530–2,040*	Torsion
F.....	1.96	.00020	1,000–4,000	Force
9 A	1.13	.00024	600–4,000	Force

In column 4 for the series marked *, no measurements were made at field strengths either above or below the limits indicated.

to intensity, are analyzed in Fig. 5. The observed values are plotted in curve A. The straight line B is drawn through the downward slope of curve A, and is assumed to represent a diamagnetic component of

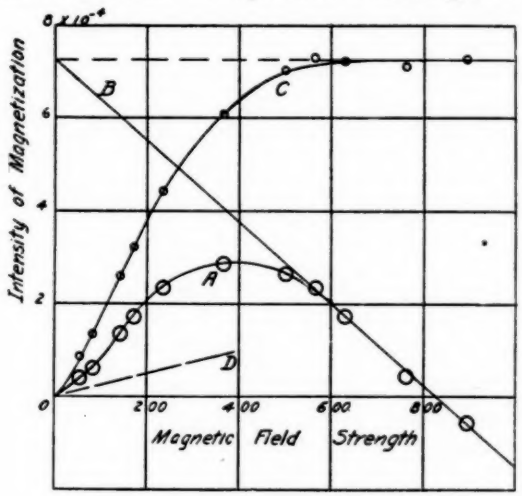

Fig. 5.

Magnetization of Silver.

Data for curves A, B, and C taken from a paper by Frivold.

Curve A: observed values of intensity of magnetization, for unit volume.

Curves B and C: the two assumed components of A, B diamagnetic, and C ferromagnetic.

Curve D: initial susceptibility of silver as observed by Shaw and Hayes, for fields below 12 gausses.

magnetization, which is added to the assumed ferromagnetic component C to give the observed values of curve A. The line OD represents an average value of the initial susceptibility of silver as observed by Shaw and Hayes[1] for fields below 12 gausses.

In this case, as also for a number given by Honda,[2] there is enough iron present as an impurity to account for the phenomenon. The behavior of this iron is, however, somewhat capricious. A specimen of silver containing but seven parts in a million of iron was found by Honda to have a diamagnetic susceptibility which increased with the field strength, whereas a specimen of cadmium containing forty-three parts of iron in a million had a constant susceptibility. In the case of hematite it seems improbable that the phenomenon can be ascribed to the presence of an iron impurity in the ordinary sense of the word, because the ferromagnetism conforms to the crystalline structure.

It is a possible inference that the normal curve for magnetization would be of this character. Such a superposed paramagnetism in iron, cobalt, nickel, or magnetite would be masked by the intense ferromagnetism of these substances. I have hunted through the accessible literature on the Heusler alloys without finding evidence one way or the other. It may be that, with relatively low intensities of magnetization of the order of magnitude met with in hematite, further evidence could be obtained. There are one or two minerals which offer promise in this direction, and I hope in the not distant future to be able to investigate this material.

II. Hysteresis.

The hysteresis losses accompanying a reversal in the direction of magnetization may be determined from torsional measurements in two ways: (1) by rotating the sphere in a uniform field, and measuring the torque at different angles, and (2) by suspending the sphere with the magnetic plane nearly parallel to the field, *i.e.*, with α small (Fig. 1) and then causing the field to increase from zero to a maximum value and then reversing, measuring again the torque as the field varies. Both methods were used, and the results from the two agree very closely.

6. *Rotational Hysteresis. Low Fields.*—In taking these values the solenoidal coils referred to in the preceding section were mounted on the rotating table and rotated about the specimen, angles being read from a divided circle carried by the coils and deflections being read by means of a telescope and scale. The set up was the same as that used in determining the component of the intensity of magnetization in the magnetic

[1] Electrician, 77: 708, 1916.

[2] Ann. d. Physik, 32: 1027, 1910.

plane, as described above. The results are shown plotted in Fig. 6, the scale deflection, which is proportional to the torque, being plotted against the angle of rotation. In Fig. 6 are also indicated the couples, reduced to absolute units of force and unit volume of specimen, these

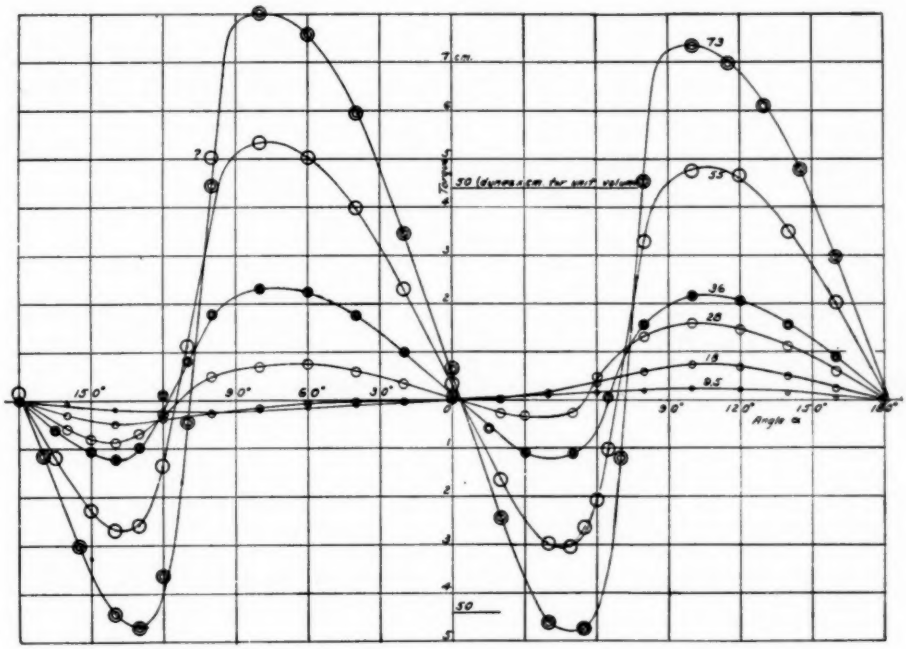

Fig. 6.

Torques at Low Fields.

Sphere *G*, Ouropreto, Brazil. Torque as dependent upon α (Fig. 1) for fields of 9.5, 18, 28, 36, 55, and 73 gausses.

values being obtained by multiplying the observed deflections by 1.90 (the torsion constant), and dividing by .1092 (the volume of the sphere). Fields above 73 gausses were not used because of the heating of the coils due to the exciting current.

Using the curves of Fig. 6, the values of the intensity of magnetization were calculated, and the values obtained are plotted in Fig. 7. The initial demagnetization was obviously imperfect, and strong hysteresis is apparent in the curves. They are likely not to be symmetrical at these low fields, though at higher fields this asymmetry disappears.

Intermediate Fields.—Rotational hysteresis curves were obtained with the same set up as was used in the magnetization curve work for fields of 76 gausses and 143 gausses (Section 5, *b*). The results of these meas-

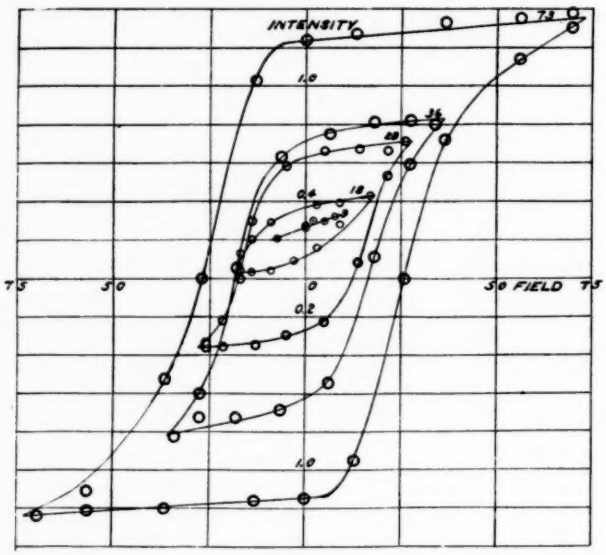

Fig. 7.

Hysteresis Loops for Sphere G.

Calculated from Fig. 6, on the assumption of the independence of the components of the magnetization along the axis and in the plane of symmetry.

urements are shown in Fig. 8, where the torques are plotted against the angle of rotation. Here the hysteresis is clearly in evidence, and the curves are nearly symmetrical. The results show the difference in the

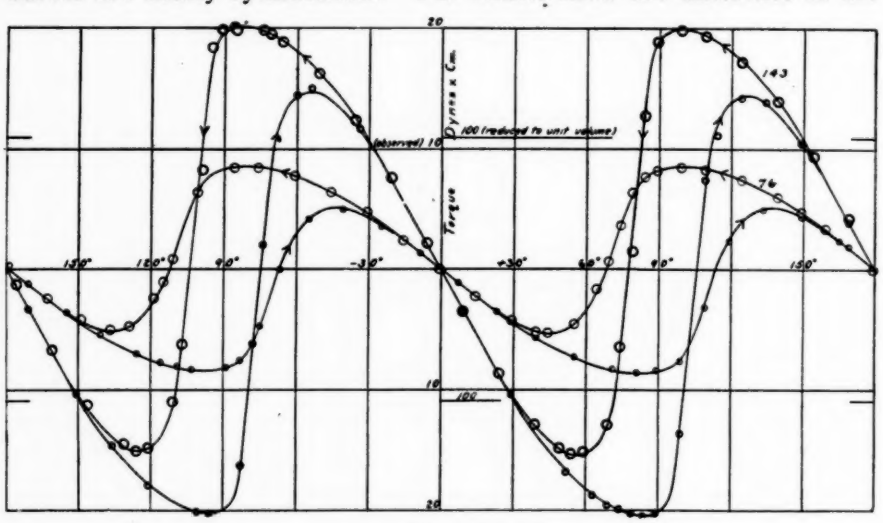

Fig. 8.

Torques at Intermediate Fields for Sphere G.

Curves taken with rotating electromagnet. Hysteresis is shown by the separation of the curve into two branches, corresponding to different directions of rotation.

torques obtained when the magnet is rotated first in the one and then in the other direction, the direction of rotation being indicated by arrow heads.

The hysteresis energy losses for the various loops of Figs. 6 and 8 are tabulated in Table IV., the cycles made with the electromagnet being designated with an (*e*). To these have been added the losses calculated from several curves with alternating hysteresis, the method of obtaining which is described in the following section. The values obtained with the solenoids are a little greater than those obtained with the electromagnet, but if this difference be allowed for, it will be seen that the hysteresis energy losses for alternating and for rotating fields are the same.

TABLE IV.

Hysteresis Energy Losses for Sphere G.

Maximum Field.	Energy Loss in Ergs per Cycle.	Type of Hysteresis.
9	Not measurable	Rotational
18	7	"
28	30	"
36	52	"
55	106	"
73	144	"
76(*e*)	129	"
143(*e*)	211	"
39	60	Alternating
80	150	"
92(*e*)	152	"
188(*e*), 238(*e*), 340(*e*), 450(*e*)	211	"

The intensity of magnetization for these various fields was obtained from Fig. 4, and the logarithms of the intensity so obtained plotted against logarithms of the energy losses. The five points for fields of 36, 55, 73, 39, and 80 gausses are found to lie on a straight line. This is Steinmetz's law of energy loss[1] in hysteresis cycles, which for this specimen, is represented by

$$\text{Energy Loss} = 0.036 \times (4\pi I)^{3.0}.$$

These constants should be regarded as rough approximations.

The hysteresis energy loss seems not to increase appreciably as the field is made larger than 143 gausses, and the loop at this field strength

[1] Trans. American Inst. Elec. Eng., Jan. 19, 1892. Ewing, Magnetic Induction in Iron, etc., p. 111.

seems to have reached its final shape, which is represented in Fig. 9. The points calculated from the data represented in Fig. 8 are marked with the larger circles.

For fields above 143 gausses the spheres were unstable when the plane of symmetry, *i.e.*, the magnetic plane, was nearly perpendicular to the direction of the field. The direction of magnetization reversed and consequently the sign of the couple due to the magnetic field also reversed, changing rapidly, and the torsion constant of the suspending wire was not great enough to hold the sphere in place. It was thus impossible with the suspension used to get readings within a range of about 25 degrees when α was nearly 90° for fields only a little stronger than 143 gausses.

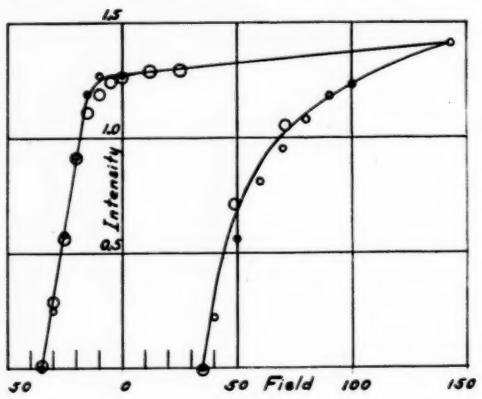

Fig. 9.

Limiting Hysteresis Loop for Sphere G.

The larger circles indicate points obtained from the torques in a rotating field of 143 gausses, Fig. 8. The smaller circles represent alternating hysteresis and the points are calculated from the curves of Fig. 10.

7. *Alternating Hysteresis.*—For this work the only change made in the experimental set-up was to attach to the rotating magnet a small mirror for the measurement of small angles with telescope and scale. The plane of symmetry of the specimen was set at a small angle to the magnetic field and the torques were measured as the field was increased to a maximum, decreased to zero, and then reversed. The maximum fields used were 92, 188, 238, 340, and 450 gausses, and the torques were measured in each case for a number of intermediate values. These torques for the four sets with maximum fields of 188, 238, 340, and 450 gausses are plotted in Fig. 10, the different sets being distinguished from one another in the plot by marking them with successively larger circles. In addition, the torques from which the magnetization curve of Fig. 4 was obtained are indicated by crosses in the branch *OB* of Fig. 10.

With decreasing field strengths (Branch *OA* of Fig. 10) and with increasing fields (branch *OB*), it appears that the magnetization follows the same path for all five sets of readings—the four hysteresis loops and the magnetization curve for which the specimen was carefully demagnetized.

On *OB* no torques are indicated for field strengths less than about 100 gausses. These torques are shown on *OCD*, to a scale ten times as great,

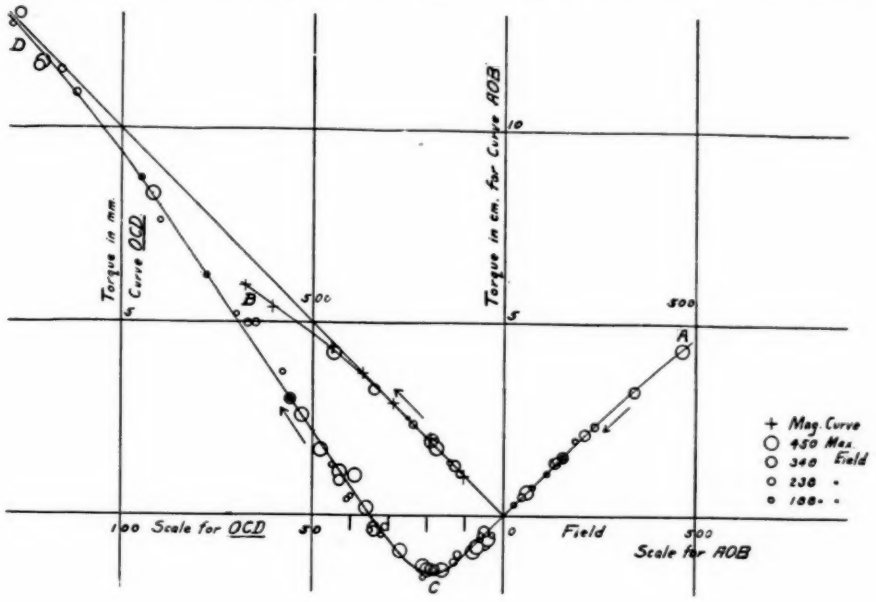

Fig. 10.

Alternating Hysteresis for Sphere G.

The angle between field and magnetic plane is here small.

The curve *AOB* represents the torques exerted as the field is changed from a maximum value in one direction to a maximum in the opposite direction, magnet stationary. Torques taken from the magnetization curve readings, Fig. 4, are shown with crosses in *OB*. The range from zero to 130 gausses is shown in the curve *OCD* to ten times the scale of *OB*.

The hysteresis loop of Fig. 9 is calculated from the curve *OCD*.

both for the deflection and field strength. Here again it is evident that the four sets of readings follow the same course, and that, within the limits of field strength here used, the hysteresis curve is independent of the maximum value of the magnetizing field. There is, then, a final hysteresis loop for hematite—at least for this particular specimen—which can be obtained from the readings plotted in Fig. 10. The smaller circles of Fig. 9 represent values calculated from Fig. 10.

It is seen that the agreement between these values and those representing the rotational hysteresis with a maximum field of 143 gausses is

good. This agreement is perhaps the most direct evidence of the independence within these limits of field strength of the effects produced by the two components of the magnetic field, the one along the axis and the other in the plane of symmetry.

It was considered desirable to test this curve at higher fields and consequently with a stronger suspension a rotational hysteresis loop was taken with a field of 1,020 gausses, and the results are plotted in Fig. 11. It is, I think, obvious that any accurate determination of the

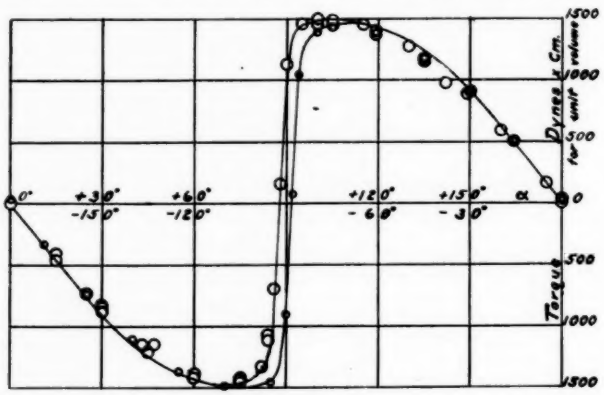

Fig. 11.

Rotational Hysteresis for Sphere G. Field of 1020 gausses.

area is impracticable, but such checks as can be applied agree with the assumption of a maximum hysteresis loop.

The areas check approximately and the coercive force checks very well. With a coercive force of 35, as shown in Fig. 9 and a field of 1,020, the width of the hysteresis loop should be just under four degrees, and the width of the loop in Fig. 11 is very nearly that.

There is no such maximum hysteresis loop within the range of field strengths which have been used for experiments with sphere 3, Elba hematite, in which the energy losses have the values of 79, 154, 195, 266, and 377 ergs for fields of respectively 76, 143, 208, 308, and 450 gausses. The question has not been tested with the *J* spheres, for which the hysteresis is certainly much less. In any case the significance of the observation in the case of sphere *G* consists in the existence of the maximum loop, and not in the area, which seems to depend upon the particular specimen used.

A hysteresis loop of maximum area has been observed by Weiss[1] in the case of pyrrhotite above 2,000 gausses, the energy loss being about 2,900 ergs per cycle, with a maximum intensity of magnetization of

[1] Journal de Physique (4), 4: 829, 1905.

about 47. This "normal" hysteresis loop he supposes to be rectangular, while in the case of hematite the normal loop would be a rhomboid, with an energy loss of about 211 ergs and having the curve cross the axis at about $I = 1.50$.

Baily's[1] calorimetric study of the hysteresis energy loss at high excitation in soft iron indicates that for alternating fields the hysteresis energy loss approaches a maximum as the iron approaches saturation. The limiting energy loss with the specimen of iron used by Baily was about 30,000 ergs per cycle.

Attention should be called, to the fact that the shape of the curve in Fig. 11 does not entirely bear out the assumption of the independence of the effects produced by the two components of the energizing field. The experimental curve changes less abruptly in the neighborhood of 90° than would be required by the calculated values.

According to Fig. 9, the residual intensity of magnetization should be 1.27, and with a field of 1,020 the torque at 90° should be, therefore, about 1,300 dynes × cm., when reduced to unit volume. The two observed points at 90° are 900 and 1,125 (Fig. 11), which are appreciably lower than the independence of the two field components would lead one to expect.

These differences are puzzling, in view of the excellent agreement exhibited in Figs. 4, 9 and 10. A possible explanation is that the assumption of the independence of the two components of the field is not strictly justifiable—at least not in the immediate neighborhood of the axis of the crystal. It is, also, possible that the differences are caused by the oscillation of the sphere in the field, arising from the elastic suspension and the step by step rotation of the magnet.

At a little distance from the axis, the agreement is better. In Fig. 11 the maximum torque is about 1,500, which would correspond to a component of magnetization perpendicular to the field of about 1.47, when the angle is about 75 degrees. Such a component would indicate an apparent intensity in the magnetic plane of about 1.515 for a field of about 264 gausses. Adding now .000117 × 264, to allow for the axial paramagnetism, the result is 1.55, which is close to the magnetization curve of Fig. 4.

III. Magnetization Oblique to the Principal Plane.

8. The product of the magnetic field strength multiplied by the component of the intensity of magnetization perpendicular to the field gives the torque exerted on a magnet of unit volume set obliquely to the

[1] Phil. Trans., A, 187: 715, 1896. Electrician, Nov. 22, 1895.

magnetic field. Consequently if the torques represented in Fig. 11 be divided by 1,020 the result will be the component of the intensity of magnetization perpendicular to the field. If the mean of the two branches of this curve be taken, the effect of hysteresis will be at least partially eliminated. This was done.

The perpendicular component so obtained is shown in curve *B* of Fig. 12, for values of α from zero to 90 degrees, the zero direction being

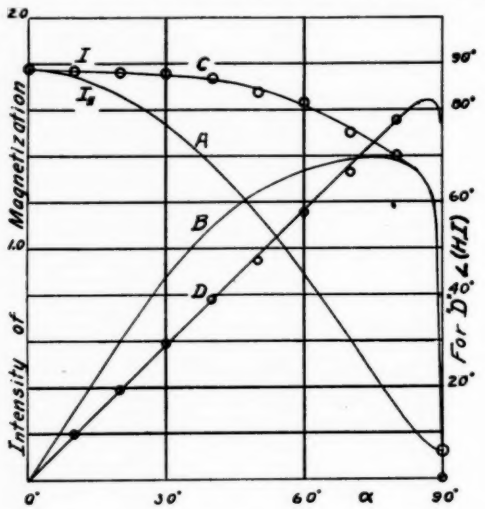

Fig. 12.

Magnetization Oblique to the Axis.

Sphere *G* in a field of 1020 gausses.
Curve *A*: component of intensity parallel to the field.
Curve *B*: component of intensity perpendicular to the field.
Curve *C*: resultant intensity of magnetization.
Curve *D*: angle between directions of field and of magnetization.
Scale for *D* shown to right of figure.

the direction of the principal plane. From Fig. 3 of the author's previous paper (loc. cit.) the component of magnetization parallel to the field may be determined by interpolation, and this is shown in curve *A* of Fig. 12. From these two sets of values the resultant magnetization (curve *C*) and the angle between *H* and *I* (curve *D*) were determined in steps of 10 degrees, giving the points indicated. Neither *B*, *C*, nor *D* are accurately determined near 90 degrees and the curves in that region indicate only that the values change rapidly with α.

The author is indebted to Professor E. F. Stimpson for some very timely assistance in the experimental work.

BLAKE PHYSICAL LABORATORY,
UNIVERSITY OF KANSAS,
August 31, 1919.

UNIPOLAR INDUCTION.[1]

BY W. F. G. SWANN.

SYNOPSIS.

Unipolar Induction; Theory of the Uniform Motion of a System of Amperian Current Whirls.—The paper forms an inquiry as to the extent to which the so-called "Moving Line Theory" is the equivalent of the Maxwell-Lorentz theory. The latter gives the electric force on a fixed unit of charge in the form:

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{grad}\,\psi,$$

where U is the Maxwellian vector-potential, and ψ is the electrostatic potential.

At first sight it might appear that ψ would be zero in the case of a *magnet which was uncharged when at rest.* It is shown, however, that the mere fact of the absence of resultant force upon a charge which accompanies a magnet in uniform rectilinear motion requires a rearrangement of electric density in the amperian whirls (or their equivalents) which constitute the molecular magnets; *and, this rearrangement is such as to endow the magnetic doublets with the properties of electric doublets as well,* the axes of corresponding magnetic and electric doublets being perpendicular to each other. These electric doublets result in a *finite value for* ψ. *It appears that, when* ψ *is included in this way, the Maxwell-Lorentz theory is the equivalent of the "Moving-Line Theory"* for the case of a single magnetic doublet in uniform motion.

Application to Rotating Magnetic Systems, and Relation to the "Moving Line Theory"; the Part Played by the Motional Intensity.—The above considerations may be extended to the case of a magnetic doublet (amperian current whirl) revolving about an axis parallel to its own and outside itself. In such application *we must, however, calculate the motion of the lines of magnetic induction as though they partook only of the translatory motion of the doublet in its orbit,* and not of its rotation about its own center. In the form in which the theory is usually applied to problems of rotation, the motion attributed to the magnetic lines is the same as if the lines from the elementary whirls were pinned down at the places where they pass through the axis of rotation of the system. In other words, the motion of the lines is calculated as though it partook of the rotational velocity of the *frame* of the whirl about its own center as well as of the translatory velocity of the whirl in its orbit. Bearing these considerations in mind, we may extend the idea to a system of amperian whirls, such as a magnet rotating in bulk.

If, in the case of a *conducting magnet,* we for the moment ignore the motional intensity which would act upon a unit of charge in the conductor as a result of its motion in its own magnetic field, the field due to the electric doublets would result in a distribution of electricity throughout the conductor of such a nature as to annul the electric field at all internal and external points. The *motional intensity* then superposes upon this condition an *electrostatic distribution* which is thus the *sole representative of the ultimate effect.* In general, this electrostatic distribution results in a potential other than zero at the axis of rotation, so that *if this axis is*

[1] Presented at the meeting of the American Physical Society, October 11, 1919.

earthed, an additional charge will come to the system, and this charge will also contribute to the electric field. In the "*Moving Line Theory*" *as ordinarily applied*, the electric intensity acting upon a unit of charge at rest turns out to be equal and opposite to the motional intensity acting upon the unit of charge as a result of its motion in company with the material of the rotating system, and it therefore results that, *on this theory, there is no electrostatic distribution in the rotating body.*

Discussion of Experimental Results Heretofore Obtained.—Recent experiments upon the subject of Unipolar Induction are discussed in terms of the conclusions arrived at in the paper. One of the main conclusions reached is to the effect that, in *all experiments where a null effect was obtained, the nature of the apparatus was such as to shield off any effect which might legitimately have been expected.*

Calculation of Effects for a Rotating Sphere.—The electrical field of a uniformly rotating, and uniformly magnetized sphere is calculated and discussed, under various conditions, *axis insulated, axis earthed, inductively magnetized, and permanently magnetized.*

Magnetic Doublet Rotating about an Axis Passing through its Center, and Perpendicular to its Own Axis.—It is shown that the electric field may be calculated for this case from the "Moving Line" law provided that the doublet is split into *two imaginary poles* and the lines for each are treated separately. It is necessary, however, to make the calculation on the basis that the *lines from the poles partake only of the linear velocity of the poles in their orbits*, and not of the rotational velocities about the centers of rotation. *In this form, the "Moving Line" law is applicable to a magnetic doublet moving in any arbitrary manner* so long as the velocities are not comparable with that of light.

INTRODUCTION.

CONSIDERING any magnetic system which, when at rest, is uncharged, the question primarily involved in the so-called problem of unipolar induction is this: Is it possible to calculate the electric field produced at an external point by uniform rotation, or for that matter by any kind of uniform motion, by supposing the lines which represent the magnetic induction B to be rigidly attached to the system, and by utilizing the expression

$$E = -\frac{1}{c}[V \cdot B] \qquad (1)$$

where V is the velocity of the line perpendicular to its length, and c is the velocity of light.

Such a law is by itself *sufficient* to predict Faraday's law that the electromotive force integrated around a circuit shall be equal to the rate of change of induction through it; but, without further specification of the laws of electromagnetism, its *necessity* is not obvious. Most practical problems are concerned only with the electromotive force integrated around a circuit; and, for these, the relation (1) would lead to the correct result even though it might be erroneous as applied to obtain the field at a single point.

The class of experiments which have usually been associated with attempts to decide the validity of (1) are those in which a symmetrical

system, such as a cylindrical bar magnet, is caused to rotate about its axis of magnetization, and devices are employed for the purpose of detecting any electrical field which may be generated in the vicinity of the magnet. Some of the most recent work on this matter has been done by E. H. Kennard,[1] S. J. Barnett,[2] and G. B. Pegram.[3] The symmetry of the problem is such that there is no rate of change of magnetic flux through any circuit, so that, even if the electric intensity were zero everywhere, Faraday's law would not be violated in this case.

A perusal of the literature on this subject suggests that many physicists have a feeling that there is a fundamental element of uncertainty as to what should happen in experiments of this kind; that there is, in fact, a question to which our electromagnetic scheme has no answer. This attitude may perhaps arise from a feeling that since Faraday's laws only expresses the field integrated round a circuit, the electromagnetic equations may not possess the power to dissect out the integral into its component elements. Such would indeed be the case were we entirely dependent upon Faraday's law; but, the electromagnetic theory involves another set of circuital relations besides Faraday's law, and these, combined with Faraday's law, enable the equations to be integrated so that the value of E is determined at each point. This immediately suggests the power of these equations to answer any problem as to the value of E at a point, provided that the problem is stated with sufficient definiteness. It is of interest to inquire therefore as to the extent to which the recognized electromagnetic theory has the power to provide an answer to questions of this kind.[4] It will conduce to clarity to first formulate the general bases which the electromagnetic theory provides for the discussion of the problem. It is desirable to do this in some detail since such confusion as arises comes often from a misinterpretation of the exact meaning of the symbols and definitions occurring in the theory.

[1] E. H. Kennard, Phil. Mag., 23, 937, 1912: 33, 179, 1917.

[2] S. J. Barnett, PHYS. REV., 35, 323, 1912.

[3] G. B. Pegram, PHYS. REV., 10, 595, 1917.

[4] I discussed certain aspects of this matter at the Washington meeting of the American Physical Society on April 21, 1916, in connection with a paper by Prof. G. F. Hull, on "The E.M.F. Generated by the Rotation of a Cylindrical Magnet about its Axis." Subsequently, at the meeting of the Physical Society in October, 1916, and in PHYS. REV., 10, 595–600 1917, Prof. G. B. Pegram expressed, quite independently, conclusions almost identical with those previously expressed by myself. A similar idea is also expressed on page 1,160 of a "Report on Electromagnetic Induction" by Prof. S. J. Barnett, presented at the joint meeting of the American Physical Society and of the Institute of Electrical Engineers, October 10, 1919. I mention this not of course as a matter of priority, but because, in what follows, I shall find it necessary to disagree to some extent as to the completeness of the conclusions drawn by Professors Pegram and Barnett, and incidentally therefore with those I had formerly expressed; for, although they represent an important part of the matter, they form by no means the whole story.

GENERAL FUNDAMENTALS CONCERNED.

With electric and magnetic quantities in Heavisidean electrostatic and electromagnetic units respectively, the two sets of circuital relations are, in the case of free æther:

$$\frac{1}{c}\left(\rho u + \frac{\partial E}{\partial t}\right) = \text{Curl } H \tag{2}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t} = \text{Curl } E \tag{3}$$

and, in addition, we have:

$$\rho = \text{Div. } E. \tag{4}$$

An additional relation Div. $H = 0$ is simply added to exclude, from the problems we are interested in considering, all those in which the magnetic field arises from ordinary scalar potentials, *i.e.*, from distributions of hypothetical magnetic "charges."

H is the force on a fixed Heavisidean unit magnetic pole, and E the force on a fixed Heavisidean electrostatic unit of charge. It is important to observe that though E sometimes arises totally from electrostatic considerations and sometimes from the existence of changing magnetic fields as well, the relation (4) is universally true and is, in fact, the definition of ρ which, without some such definition, would have no meaning in the case of moving charges.

There are two forms in which (2) and (3) are customarily integrated. In the first form, which corresponds to that adopted by Maxwell, we have:

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{Grad } \psi, \tag{5}$$

where

$$\psi = \frac{1}{4\pi}\int\int\int \frac{\rho}{r} d\tau \tag{6}$$

and

$$U = \frac{1}{4\pi c}\int\int\int \frac{(\rho u + \dot{E})}{r} d\tau. \tag{7}$$

Here ψ is the true electrostatic potential of the charges. It is the potential which the various volume elements of charge would exert at the time in question if, at that instant, they were suddenly brought to rest and pinned down in the positions which they happened to occupy. The vector potential U is not expressed entirely in terms of the motions of the charges, but involves also an integration of $\partial E/\partial t$ throughout space.

Div. U is zero at all points so that the density ρ, which is equal to Div. E is also minus the divergence of Grad ψ.

In the second or Lorentzian form, we have:

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \text{Grad } \varphi, \tag{8}$$

where

$$\varphi = \frac{1}{4\pi}\int\int\int \frac{[\rho]}{r}\, d\tau \tag{9}$$

and

$$A = \frac{1}{4\pi c}\int\int\int \frac{[\rho u]}{r}\, d\tau. \tag{10}$$

Here, however, the square brackets are to be taken as indicating that, in seeking the value of φ or A at the time t, we must not put in the values of ρ and u in the various volume elements at the time t, but to each volume element we must assign the value which existed there at the time r/c earlier than t. Thus φ is no longer the electrostatic potential in the sense that ψ is. Div. A is not in general zero, so that the density is not given in general by Div. Grad. φ. The two formulas (5) and (8) are merely different ways of expressing the same quantity E. The Lorentzian form has in some cases the advantage in that it gives A and φ, and consequently E, explicitly in terms of the positions and velocities of the charges without invoking the knowledge of E at all points, as a direct determination of U would require. In both cases H is of course the curl of the corresponding vector potential.

E is the force on a unit charge fixed with respect to the observer; and, it is not always realized that neither the circuital relations (2) and (3), nor their analytical equivalents (5) or (8) have the power to tell us anything at all about the force on a charge which moves relatively to the observer. The force on a moving charge is $E + [V \cdot H]/c$, where the vectors refer to the parts not contributed by the charge itself;[1] but, in giving the name "Force" to this vector, we do not endow it with any properties which tell us in what way this so-called force influences the motions of the charges. The assumption which is made by Lorentz is that the electron moves in such a way that the force (as defined above) which it exerts on itself as a result of the motion is equal and opposite to the force which the external field exerts upon it. The expression for the former can be developed in terms of the acceleration and higher derivatives of the velocity of the electron. It turns out that the term proportional to the acceleration is the most important of all the terms

[1] The division of a field into two parts, that due to a certain charge, and that due to the remainder of the charges is a matter of definition. The field of a given charge is *defined* as the portion obtained by utilizing the parts of φ and A contributed to the integrals by the charge in question.

in the case of many types of motion; and, by giving the name "Mass" to the coefficient of the acceleration, the assumption of Lorentz becomes thrown into the familiar form "Force equals Mass multiplied by Acceleration." It is on account of the act of *forcing* the Lorentzian assumption into this form that we arrive at such apparently artificial concepts as "Longitudinal" and "Transverse" mass.

The assumption of Lorentz is not derivable from the circuital relations. Such theoretical basis as it has rests upon the fact that, neglecting certain objections which it is not the purpose of this paper to discuss, it becomes predicted by the same application of the Hamiltonian principle to the problem of the electromagnetic field as, with the proper choice of the Lagrangian function, brings out the Faraday law. In the case of a single coördinate, but only in this case, it can be deduced from the principle of the conservation of energy as might be expected since, for the case of a single coördinate, the Hamiltonian Principle is the analytical equivalent of the principle of the conservation of energy.

It is convenient to speak of the contribution $[V \cdot H]/c$ to the force acting upon an element of charge as the "Motional Intensity," so that the total force F acting upon the unity charge is the sum of the electric intensity E which would act upon the charge at rest, and the motional intensity. Thus, using (5) we have:

$$F = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{Grad. } \psi + \frac{1}{c}[V \cdot H], \tag{11}$$

where U and ψ have the meanings defined in (6) and (7). In place of U and ψ we may of course use the Lorentzian potentials if we choose.

It must be distinctly understood that there is no ambiguity of meaning depending upon considerations as to whether the magnetic lines do or do not move. H is the force which would be experienced by a fixed magnetic pole; V is the velocity of the charge upon which we desire the force, and is measured relatively to the fixed observer. The u which occurs in the expression (7) for U refers, at each point of space, to the velocity of the electricity at that point. Further, in evaluating U, ψ and H, we must omit from our integrals the parts contributed by the charge whose motion we desire to study.

Motion of Molecular Magnet or Amperian Current Whirl.

It is fairly obvious, and may be seen rigorously by a glance at (5), (6) and (7), that the electric intensity at any point may be expressed as a sum of contributions from each element of the space. It is therefore of interest to begin by studying the effect of an infinitesimal circular amperian whirl (taking place around an infinitesimal anchor ring for

example), and then discuss the field which will arise by the superposition of an infinite number of such whirls which together may be supposed to make up a magnetized body. To fix our ideas we shall suppose the whirl to be one of negative electricity; and, in order to avoid as far as the analysis will permit us, the complications arising from the portions of the field due to mere electrostatic distributions of electricity, we shall suppose that, on the top of our anchor-ring whirl, we superpose a uniform distribution of positive electricity which does not participate in the rotary motion of the whirl, but which is equal in total amount to the negative charge in the whirl.

The question which we wish to decide is whether, or under what conditions, the electric field due to the motion of such a whirl, and which is strictly given by (5) for example, may also be calculated, at any rate to a first approximation, from the moving magnetic line idea on the assumption that the lines are rigidly attached to the frame of the whirl. If such a method of calculation is possible then, by superposition of the effects, we may apply the moving line method to the motion of a magnet in bulk. In doing this, however, we must be careful to apply it to the motion of the field of each amperian whirl separately, and this will mean that in many cases, as for example in estimating the electric field at a point on the plane passing through the center of a rotating magnet and perpendicular to the axis of rotation, the lines contributed by the various elements of the rotating magnet will be moving in all sorts of directions, some indeed moving in diametrically opposite directions, for the portions of the surface of the magnet at the two ends of a diameter move in opposite directions. It is only in some such form as this that the "Moving Line Theory" could ever hope to represent the true state of affairs. Any other view would make the theory inconsistent with itself; for, if it is to apply to the magnet in bulk, it must manifestly apply to the constituent elements of which the magnet is composed.

As an introduction to our problem, we shall first consider a case of a whirl in uniform rectilinear motion.

Amperian Whirl in Uniform Rectilinear Motion.—Suppose that the axis of the amperian whirl is inclined to the axis of x, and that the rectilinear motion takes place parallel to the axis of x with velocity v. The field E is given by (5), and it might be thought at first sight that we should be justified in neglecting the term Grad. ψ, involving the purely electrostatic potential, on the ground that we superposed on our negative whirl a positive distribution which, as regards electrostatic potential, just cancelled the negative distribution when the whirl was without rectilinear velocity. Such an assumption would be unjustified, however,

for it would lead us among other things to the relation:

$$E_x = -\frac{1}{c}\frac{\partial U_x}{\partial t},$$

so that, unless U_x were zero, there would be a component of E in the direction of motion, a conclusion which no one would be prepared to admit, and which would violate the principle of relativity to an extent measured by its whole value. For, the only reason that a moving observer fails to realize that the field which he measures is changed by the motion is that the term $[V \cdot H]/c$ in equation (11) just cancels the change in E produced by the motion of the system, and, the term $[V \cdot H]/c$ has obviously no component parallel to the direction of motion. That the vanishing of E_x does not depend upon the vanishing of U_x may readily be seen by putting E_x zero in (7), which then leaves us with a perfectly definite and finite expression for U_x.

In fact, if we are to have E_x zero, as relativity considerations require, we must admit the existence of a value of ψ determined by:

$$-\frac{1}{c}\frac{\partial U_x}{\partial t} - \frac{\partial \psi}{\partial x} = 0$$

or since

$$\frac{\partial}{\partial t} = -v\frac{\partial}{\partial x},$$

$$\frac{v}{c}\frac{\partial U_x}{\partial x} = \frac{\partial \psi}{\partial x}.$$

The arbitrary function of y and z which arises on integration of this equation must be zero as otherwise U_x or ψ would be finite for infinite values of x. Thus

$$\psi = \frac{v}{c}U_x \tag{12}$$

or, remembering that E_x is zero, and using (7) we have:

$$\psi = \frac{v}{4\pi c^2}\int\int\int \frac{(\rho_0 v - \rho_1 u_x)}{r}\, d\tau, \tag{13}$$

where, in terms of measurements by a fixed observer, ρ_0 is the density of the positive electricity, ρ_1 the density of the negative distribution in the moving system, and u_x the x component velocity of the negative electricity.

If i_x represents the x component of the current density as measured by a fixed observer, equation (13) shows that the rectilinear velocity has resulted in the creation of a resultant charge density given, at each

point of the anchor-ring, by:

$$\Delta\rho = \frac{vi_x}{c^2}. \tag{14}$$

Utilizing in (5) the value of ψ given by (12), and remembering that $H = \text{Curl } U$, we have:

$$E_y = -\frac{1}{c}\frac{\partial U_y}{\partial t} - \frac{v}{c}\frac{\partial U_x}{\partial y} = \frac{v}{c}\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right),$$

$$E_z = -\frac{1}{c}\frac{\partial U_z}{\partial t} - \frac{v}{c}\frac{\partial U_x}{\partial z} = \frac{v}{c}\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right),$$

so that:

$$E_x = 0, \qquad E_y = \frac{v}{c}H_z, \qquad E_z = -\frac{v}{c}H_y \tag{15}$$

or, in vector notation,

$$E = -\frac{[v \cdot H]}{c}. \tag{16}$$

Equation (16) is obviously the expression of the "Moving Line Theory" for the case of uniform rectilinear motion. In a sense, it is in more exact accord with the "Moving Line Theory" than are the usual statements of that theory; for, in (16) H represents the true value of the magnetic field as modified by the motion, and the result is true to any order of magnitude. As will be seen, moreover, from the analysis, the result is not limited to the particular amperian-whirl system which we have chosen as the basis of our discussion, but is true for example where the amperian whirls are replaced by electrons revolving about positive centers. The amperian whirl is however, the easiest system in terms of which to visualize the phenomena, and we shall continue to discuss matters in terms of it.

As may be seen from (14), ψ arises from a crowding together of the negative electricity on the side of the whirl on which the x component of the rotational velocity coincides with the rectilinear velocity of the whirl, and a thinning out of the negative density on the other side. In fact, the electrostatic potential corresponds, at large distances, to that of an electric doublet. The rearrangement of the density of the negative electricity is not cancelled by a similar rearrangement of the positive because, as shown by (14), this arrangement depends upon the velocity relative to the moving center as well as upon the velocity of the moving center itself.

A physical picture of the origin of ψ may readily be obtained. For suppose that our elementary whirl was originally at rest, as a whole, and without charge, and that it is now in rectilinear motion in a direction

perpendicular to the plane of the paper, with its axis parallel to the plane of the paper, Fig. 1. Remembering that the magnetic lines pass upwards, for example, as they thread through the whirl and downwards as they return on the outside, everyone will admit that, at the surface of the anchor ring there will appear an electric field which will be in opposite directions at points A and B on the interior and exterior surface of the anchor ring. This of course means that there is a resultant electric flux from, and consequently a charge on, the part AB of the anchor ring. It is easy to see from the directions of the magnetic and electric forces that the sign of the charge at the place CD is opposite to that of the charge at AB, so that these two places contribute to the formation of an electric doublet, equilibrium being secured by the motional intensity.

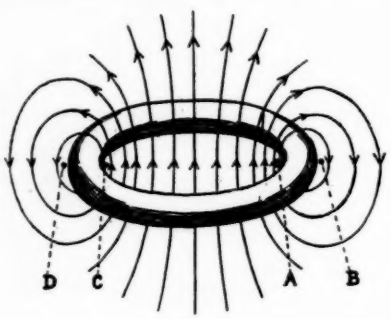

Fig. 1.

It thus appears that, even in the case of a magnetic doublet which is electrically neutral when at rest, the electric field resulting from the motion of the doublet depends partly upon an electrostatic potential, and is not determined entirely by the time rate of change of the vector potential. Further, by superposition of the contributions from the various elementary molecular magnets we shall find that this effect persists in the case of the rotation of a magnet in bulk. It is true, as we shall see later, that, in the case of a *conducting* magnet, the electrostatic potential causes a redistribution of the free electricity in the magnet, of such a nature as to annul its effect. The secondary electrostatic potential arising in this way is, however, of an origin quite different from that of the primary potential due to the electric doublets, as is also the potential of the electrostatic distribution which results from the motional intensity produced by the rotation of the conductor in its own magnetic field.

In the case of a symmetrical rotating magnetized system, the vector-potential is independent of the time; but, as we have seen above, this is not a sufficient criterion for the prediction of absence of electric field

as a result of rotation. It is in this respect that I find it necessary to differ from the conclusions expressed by Professors Pegram and Barnett,[1] and incidentally therefore from those which I myself had expressed at an earlier date.

It may be remarked that the results which we have deduced as to the creation of an electric doublet in an amperian whirl as a result of imparting rectilinear motion to the whirl may be deduced by a direct application of the transformations of the theory of relativity, as may also the equivalence of the Lorentzian theory and the "Moving Line Theory" for this case.

Thus, if ρ' is the density at a point in any system, as measured by a fixed observer and the system be set in rectilinear motion with velocity v, ρ' will be the density as measured by an observer participating in the motion. But the density ρ as measured by a fixed observer will now be given by[2]:

$$\rho' = \beta\left(1 - \frac{vu_x}{c^2}\right)\rho$$

where u_x is the x component velocity of an element of the charge as measured by the fixed observer, and β is written for $(1 - v^2/c^2)^{-1/2}$. If ρ_0' and ρ_1' refer respectively to the densities of positive and negative electricity in our amperian whirl, for example, when without rectilinear velocity, while ρ_0 and ρ_1 refer to the corresponding quantities as measured by a fixed observer when the system has rectilinear velocity v, we have:

$$\rho_0' = \rho_1', \qquad \rho_0' = \frac{\rho_0}{\beta}, \qquad \rho_1' = \beta\left(1 - \frac{vu_x}{c^2}\right)\rho_1,$$

so that:

$$\Delta\rho = \rho_0 - \rho_1 = -\frac{vu_x}{c^2}\rho_1 + \frac{v^2}{c^2}\rho_0 = \frac{vi_x}{c^2},$$

which is the same result as that obtained in (14).

It must not be thought that we have *proved*, from the purely electromagnetic considerations contained in equations (5) to (7), that there is a redistribution of density in our amperian whirl as a result of its rectilinear motion. The theory of relativity is not deducible from electromagnetic considerations alone. What has been shown is that, if the theory of relativity is to hold, the density distribution must come about in the manner we have stated. When, in deducing (14), (15), and (16), we assumed that E_x was zero, we of course made an assumption which, for this purpose, was the equivalent of the theory of relativity. If we do not assume something such as the principle of relativity in addition to

[1] See note 5 of this paper.

[2] See for example, O. W. Richardson's Electron Theory of Matter, p. 307.

our electromagnetic scheme, we have no basis for forming a conclusion as to what happens to our whirl as a result of rectilinear motion. If we were to *assume* the negative and positive densities in our whirl to be unaltered at each point, so that one offset the other, as regards electrostatic field, we should not arrive at the "Moving Line Result." We should, moreover, predict the very astonishing conclusion that rectilinear motion results in the appearance of a component of E parallel to the line of motion, a result quite inconsistent with the theory of relativity.

In addition to the result (14), we may of course deduce also the results (15) by a direct application of the relativity transformations. For if symbols with and without dashes correspond respectively to quantities measured by a moving and by a fixed observer, the relativity transformations are:[8]

$$E_x' = E_x, \qquad E_y' = \beta\left(E_y - \frac{v}{c}H_z\right), \qquad E_z' = \beta\left(E_z + \frac{v}{c}H_y\right)$$

The symbols with dashes correspond also to the quantities which would have been measured by a fixed observer before the system was set into rectilinear velocity. The transformation equations thus tell us that if a system is devoid of electric field when without rectilinear motion, the impartation of uniform rectilinear motion will result in the appearance of an electric field given by:

$$E_x = 0, \qquad E_y = \frac{v}{c}H_z, \qquad E_z = -\frac{v}{c}H_y,$$

which are the same as equations (15).

This method is, of course, more general than that adopted above, and shows at once that the results are not limited to the particular case of an amperian whirl. The purpose of this paper is, however, to dissect out the physical processes at work in as clear a manner as is consistent with a rigorous treatment; and, in this respect, the more laborious development we have given may perhaps be more convincing than a direct appeal to the relativity transformations.

Moment of the Electric Doublet.—It has been shown that the part of the electric field of the moving amperian whirl determined by ψ is the same as would be produced by an electric doublet. To the extent that the magnetic field of the moving whirl is unaltered by the motion, *i.e.*, to a high approximation, the whirl is equivalent in its magnetic effect to a magnetic doublet; and, it is of interest to determine the relation between the moments M and N of the magnetic and electric doublets respectively.

For the value of U_x due to a magnetic doublet of moment M, and directional cosines l, m, n, we have[1]

$$U_x = -M\left(m\frac{\partial}{\partial z} - n\frac{\partial}{\partial y}\right)\cdot\frac{1}{r}$$

so that, from (12), we have:

$$\psi = -\frac{v}{c}M\left(m\frac{\partial}{\partial z} - n\frac{\partial}{\partial y}\right)\frac{1}{r} = \left(-\frac{v}{c}M_y\frac{\partial}{\partial z} + \frac{v}{c}M_z\frac{\partial}{\partial y}\right)\cdot\frac{1}{r},$$

which shows that ψ is the same as would result from an electric doublet of moment given by:

$$N_z = -\frac{v}{c}M_y, \qquad N_y = \frac{v}{c}M_z.$$

Thus, employing vector notation, we may write, quite generally:

$$N = -\frac{[v\cdot M]}{c}. \tag{17}$$

Rotation of American Whirl about its own Axis.—Rotation of an amperian whirl about its own axis of rotation can not give rise to any electric field; for, such rotation is simply equivalent to adding (algebraically), equal velocities to the positive and negative electricity in the whirl, so that the current density is unchanged. Neither on the basis of physical intuition, nor on the more exact basis of equations (5) to (7), should we be justified in expecting any electric field as a result of such a phenomenon. For, unless rotation of the whirl gives rise to a change in its constitution, such as to result in a finite value for ψ, the most that (5) could predict is:

$$E = -\frac{1}{c}\frac{\partial U}{\partial t},$$

which shows that E is zero in the steady state. Change of constitution of the whirl as a result of mere rotation, unlike that following from rectilinear motion, is not called for by any known fundamental considerations, and is, moreover, highly improbable from considerations of symmetry.

The above arguments may be applied, even with greater ease, to the case of the rotation of current-carrying solenoids, such as have occasionally been used in experiments on unipolar induction. No electrical effects are to be expected as a result of the rotation of such a solenoid about its own axis.

[1] See, for example, J. H. Jeans' Electricity and Magnetism, p. 394.

APPLICATION TO THE CASE OF SYMMETRICAL MAGNETIZED SYSTEMS ROTATING ABOUT THE AXIS OF SYMMETRY.

We may look upon the field due to a piece of magnetized material as made up of the fields of the innumerable magnetic doublets, or amperian whirls, of which it is composed. We may confine ourselves to the case where the axes of the elementary whirls are parallel to the axis of the rotating system as a whole (*i.e.*, to the axis of magnetization), since effects due to the motions of the components of the whirls perpendicular to the axis of magnetization must necessarily cancel out, since such components will be as often directed in one way as in the opposite way. Any doubts as to the validity of this reasoning may be removed by a subsequent reconsideration of the effects of these components in terms of the ideas to be developed in the remainder of the paper.

By way of introduction, let us first consider the case of a single amperian whirl which revolves about some axis outside itself, but parallel to its own axis. In the first place, the acceleration of the whirl as a whole towards the center of its orbit may result in a change in its actual constitution of a kind analogous to that which we have shown to result from uniform rectilinear motion. For rotation, however, we have no theory of relativity which will enable us to form a prediction for this case. Indeed, it is unlikely that there could be any theory which would predict the effects of rotation independently of the knowledge of the atomic structure of the particular material concerned. To realize the force of this contention, we have only to remember that a disc in uniform rotation will, in general, alter its dimensions by centrifugal force to an extent depending upon its elastic properties, *i.e.*, to an extent depending upon the molecular and atomic structure of the particular substance we happen to be dealing with; while, for uniform rectilinear motion, the theory of relativity predicts a contraction which is the same for all substances. If, however, we make the assumption that the distribution of electricity density in the whirl is altered by the *orbital* motion only to an extent determined by the *linear* velocity of the whirl we can proceed. Such an assumption would probably be readily accepted by most people as very reasonable; we must remember, however, that it is ultimately an *assumption*, and only to the extent that it is true will the solution of our problem be possible in the form in which we shall attempt it. It will appear later, however, that, as regards the external field due to a symmetrical, rotating, *conducting* magnet, there is no ambiguity arising from uncertainties as to the effect of rotation on the constitution of the whirl.

Even with the assumption cited above, the field of the whirl at any point, at any instant, will not in general be the same as that which would be calculated on the assumption that the whirl had been moving for an infinite time in a straight line with the constant velocity equal and parallel to the velocity which it has in its orbit at the instant concerned. Provided, however, that the velocity is not comparable with that of light, the assumption that the field may be calculated from the pure rectilinear velocity will be approximately justified. The assumption is, in fact, analogous to the so-called Quasi-Stationary Principle so frequently used in discussing the fields of moving electrons; and it involves corrections of an order of magnitude smaller than those in which we are interested here. It merely neglects the radiation field which is sent out by our amperian whirl as a result of the acceleration which it suffers in its orbit.[1]

If then we make this justifiable assumption, the results of the last section show that the electric field which the Maxwell-Lorentz theory predicts as a result of the motion of the whirl is exactly the same as that which we should calculate from the moving line theory provided that, in applying that theory, we exclude the part of the motion of the lines of force which would result from the rotation of the *frame* of the amperian

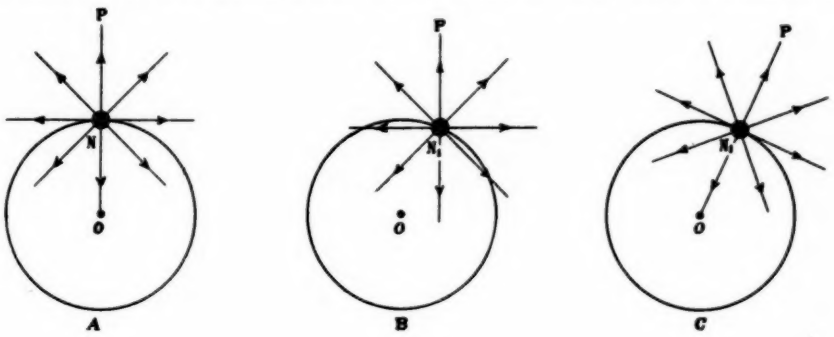

Fig. 2.

whirl about its *own* axis. In fact, in calculating the motion of the lines of force of our amperian whirl, with its axis parallel to the axis of rotation, we must not imagine the lines as attached rigidly to the frame of the whirl but rather, let us say, to a plane which passes through the center of the whirl and partakes of its linear velocity, but which remains parallel to itself during the rotation. Referring to Fig. 2, for example, where the axis of our whirl is perpendicular to the plane of the paper, where N represents the position of this axis, and O the axis of the *planetary* orbit

[1] In any case, when we consider the superposition of effects due to all the whirls in a symmetrical rotating body, it is easy to show that terms of this kind would balance out.

of the whirl, we see that, if the "Moving Line Theory" is to agree with the Maxwellian theory, then, on passing from the position N to the position N_1 we must calculate the velocity of a line of force NP as though it had assumed the orientation N_1P in diagram B, and not as though it had assumed the orientation N_1P in diagram C. A calculation on the latter basis would correspond to assuming the magnetic lines to partake of the rotational velocity of the frame of the whirl; and, it is in this respect that the "Moving Line Theory" as it is usually applied to rotational problems differs from the theory in the form in which it ought to be applied.

Thus, let us consider the case of a symmetrical system rotating with uniform angular velocity about its axis of magnetization; and, in order to avoid complications which will be presently discussed and elucidated, let us suppose that our system is non-conducting, and of unit specific inductive capacity. A convenient system to think of is two cylindrical bar magnets with axes in line, with opposite poles facing each other, and separated by a space. Since the effects of the individual amperian whirls are additive, the field due to each may be calculated on the basis of the "Moving Line Theory" as properly applied, and the resultant may then be obtained by summation. We cannot apply the "Moving Line Theory" to the resultant magnetic field, however, as is usually done when the line of force passing through the axis of rotation is supposed to be stationary, and the other lines of force are supposed to revolve around it with constant angular velocity. Such an application results in just what we should obtain were we to suppose that, in the case of each whirl, the particular line of force which happened to pass through the axis of rotation of the system was pinned down at the axis. In other words, the usual application of the theory corresponds to assuming the lines of force of the amperian whirls to partake of the *rotational* velocity of the *frame* of the whirl about its own axis, as well as of its rectilinear velocity in its orbit; and, as we have seen, such an assumption would be contrary to the Maxwell-Lorentzian requirements, and indeed to physical intuition as well.

Referring to the two cylindrical magnets with axes in line and poles facing, as cited above, we see that the ordinary application of the "Moving Line Theory" would result in predicting, for the space between the poles, an electric field radial to the axis of rotation. The divergence of this field would not be zero, and we should be driven to the impossible conclusion that there was a creation of charge density in the space between the magnets, a conclusion to which a proper application of the theory could not possibly lead, since the proper theory makes Div. E zero at all points except at the whirls themselves.

In a symmetrical uniformly rotating system, the vector potential does not change with the time, so the second member of the right-hand side of (5) is the only term which contributes to E. As we have seen in the preceding section, the effect of this term is the same as we should have if each amperian whirl were replaced by an electric doublet whose axis was perpendicular to the direction of motion. In the case under discussion, the axes of the doublets would be radial; and, if the system is uniformly magnetized, the moments of the doublets will be proportional to the distance from the axis of rotation.

By Poisson's mathematical theorem relating to distributions of doublets, the effect of such an assemblage of doublets is equivalent to that of a fictitious charge distributed throughout the volume of the magnets, and an equal fictitious charge distributed over the surface, the density of the fictitious volume charge being equal, at each point, to minus the divergence of the polarization (moment per unit volume of the doublets) and the density of the surface charge being equal, at each point, to the normal component of the polarization at that point. The field as determined by these fictitious charge distributions is thus the analytical equivalent of the field which would be obtained by a proper application of the "Moving Line Theory" to a symmetrical, insulating, and uniformly rotating magnet, of unit specific inductive capacity.

Suppose now we extend these ideas to the case of a conducting system. If for the moment we ignore the last term in (11), which corresponds to the motional intensity brought into play on the electrons of the conducting magnet as a result of its motion in its own magnetic field, we are left with the field due to the electric doublets referred to above, and it will be necessary for a redistribution of electricity to take place in the conducting magnet in such a way as to annul the field therein produced by the electric doublets. At a point within the substance, the field due to a distribution of electric doublets of this kind is composed of two parts. We have a part E_1, represented by the field which would be measured in an infinitesimal cavity whose length is parallel to the polarization and whose diameter is infinitesimal compared with its length. Then, we have a part E_2, which depends upon the local effect of the doublets in the vicinity of the point concerned. The part E_1 is the part which would be given by the fictitious charges: it is the part whose line integral over a path comprising an infinite number of molecules is equal, in the limit, to the actual difference in potential between the starting and ending points, provided that neither of these points is chosen at a distance from one of the individual doublets small compared with the average distance between the doublets. The part E_2 fluctuates in magnitude, and even

in sign, from molecule to molecule. At any point, it is proportional to the polarization (moment per unit volume), the factor of proportionality being a function of the position of the point. Its average value along any line long compared with molecular dimensions is, however, zero. The field E_1 determined by the fictitious charges is consequently the only part of the total field which is operative in causing finite movements of the free electrons of the conducting material. In order to balance this part, it is necessary for a real separation of electricity to take place in the material in such a way that the real volume and surface densities just annul the fictitious volume and surface densities at each point of the material. Such a redistribution of electricity will, however, result in complete cancellation of the external field due to the fictitious charges, so that, as a result of these influences, there will be no field at any point. This idea may readily be extended to the field due to any type of rearrangement of electricity in the amperian whirl; so that, any uncertainty as to the influence of orbital rotation in altering the constitution of the whirl is without effect on the ultimate conclusion as to the absence of any external field resulting from the combined actions of these circumstances.

If we now introduce the last term of (11), representing the motional intensity, it will result in an additional distribution of electricity, such as to just cancel its effect within the material. At a point within the material where the actual magnetic field is H, the motional intensity is given by $[v \cdot H]/c$. Now the actual value of H in the material varies rapidly in direction and magnitude in the vicinities of the amperian whirls. Nevertheless, the true H, unlike the ordinarily defined H as measured in a filamental cavity, is a solenoidal vector; and, since B, the induction, is a solenoidal vector which is equal to H outside the material, its average value normal to any element of surface of dimensions large compared with the dimensions of the amperian whirls, is the same as the average value of the normal component of the true H taken over that element of surface. Moreover, this is true in whatever way the element of surface may be made to bend about between the amperian whirls. The effective value of the motional intensity is thus $[v \cdot B]/c$; and, the electrostatic distribution which results will consequently be of a nature such as to produce an electric intensity—$[v \cdot B]/c$.

It is of interest to inquire the extent to which the above considerations must be modified in case the rotating material has specific inductive capacity greater than unity. At first sight we might argue that the state we have predicted above is one which results in there being no force on an element of electricity moving with the material and that, in

consequence, there will be no *additional* polarization of the material as a result of its specific inductive capacity. Such an argument would not be justifiable, however; for, while the effect of the field E_2 representing the local effect of the electric doublets averages out when integrated over a random path in the material, it does not necessarily average out when applied at an electron in the atom itself; for, there is a definite relationship in position between the electrons of the atoms and the electric doublets which figure in the determination of E_2. Additional polarization of the medium might therefore result, even though the *average* field over any finite region of the material were zero. When everything is taken into account, however, the net result of all these considerations is this: Electric doublets of the type we have discussed are created by the motions of the amperian whirls, and motional intensity is produced as a result of the motion of the material in the magnetic field. Redistribution of charge and possibly additional polarization will take place. The result of the charge distribution, and of the two possible types of polarization (electric doublet separation) must be such as to produce within the material a resultant field equal and opposite to the motional intensity $[v \cdot B]/c$. The electric intensity within the material is thus determined partly by a real charge distribution, and partly by fictitious charge distribution; but, as regards the calculation of the fields at points outside the material, it matters not whether the charges are real, fictitious, or both. The net effect of all considerations is that the external electric field is determined by that system of volume and surface charge distributions which would give, within the material, a field—$[v \cdot B]/c$ at each point.

So far we have supposed that the rotating system is insulated, although the substance of which it is composed may be a conductor. In the case where the rotating system is not only conducting but is also kept at zero potential along its axis, a further consideration becomes involved. For the motional intensity results in an electrostatic distribution which does not of itself result in the axis of rotation being at zero potential. If the axis is to be kept at zero potential, it will be necessary for an additional charge to come to the rotating system. This additional charge will distribute itself in the same manner as it would distribute itself on a non-magnetic system of the same shape, and in a manner sensibly the same as on a similar system devoid of rotation. If we earth some point of the system other than the axis of rotation, the charge which will come to the system on this account will be such as to alter the potential of the whole system by an amount necessary to bring the point which we have earthed to zero potential.

It is important to observe that, in line with the above developments,

such electric fields as are produced in the vicinity of rotating magnetic systems result entirely from electrostatic distributions, so that *there would be no electric field in any space from which the magnetic system was shielded by an earthed conducting shield which did not participate in the rotation.* In experiments on this subject, it has been customary to employ shields on the general principle of shielding off extraneous electrostatic effects, so that, quite apart from all other considerations, one reason for the absence of any measured effects is obvious.

As ordinarily applied, the "Moving Line Theory" predicts a field due to the motion of the magnetic lines inside and outside the magnetized body, and the nature of this application is such as make the field inside the substance just equal in amount and opposite in sign to the motional intensity so that, in this form, the theory predicts no tendency for the formation of an electrostatic distribution of any kind within the magnetic system. Outside the magnetic substance the theory, as ordinarily applied, predicts an electric field of the form—$[v \cdot H]/c$, where H is the resultant magnetic intensity at the point, and v the velocity which the point would have if rigidly attached to the rotating system. Moreover, this electric field is supposed to be of such an origin that it would not be affected by the presence of electrostatic screens.

Summary of Conclusions.—It will thus be seen that the complete story of the possible origins of electric fields in systems such as we have discussed is one involving a number of considerations; and, it may be well to review these considerations as follows:

(*A*) The field at a point external to the rotating system is given by (5); and, in the case of a symmetrical system rotating with uniform angular velocity, U is independent of the time, so that Grad. ψ represents the only contribution to the electric field E in this case.

(*B*) There is, of course, no electric field as a result of uniform rotation about the magnetic axis, of a current-carrying solenoid which contains no magnetizable material participating in the rotation.

(*C*) It has been shown that an amperian current whirl will experience a rearrangement of its charge density when set into rectilinear motion, even thought, when at rest, the positive and negative charge densities compensate in such a way as to result in no electrostatic potential function for the whirl. The rearrangement of charge density as a result of the rectilinear motion is of such a kind as to give rise to an electrostatic potential which is equivalent to that of an electric doublet with its axis perpendicular to the magnetic axis of the whirl.

(*D*) In the case of an insulating, uniformly rotating system of unit specific inductive capacity, magnetized parallel to the axis of rotation,

the whole electric fields at points external to the magnetized system is that due to the electric doublets referred to under (*C*). It is equivalent to the field which would be produced by a distribution of electricity of one sign throughout the volume, and an equal distribution of electricity of opposite sign over the surface of the magnetized material.

Moreover, in the case of a rotating magnetic system, the contribution of these electric doublets to the field is the true representative of the "Moving Line Theory" when that theory is properly applied to the space outside the magnetized material.

(*E*) The "Moving Line Theory" as usually applied to the space outside the magnetized material differs from the form in which it ought to be applied to be consistent with the equations of Maxwell and Lorentz. In the theory as properly applied, after resolving each amperian whirl to an axis parallel to the axis of rotation, the motions of the magnetic lines of the individual whirls must be treated separately; and, in calculating the motion, the magnetic lines must be pictured as partaking of only the *rectilinear* velocity of the whirl in its orbit, and not of the *rotational* velocity of the frame of the whirl about its own axis. The usual application of the theory attempts to deal with the velocity of the resultant magnetic field at a point; and, in so doing, it does the equivalent of supposing the magnetic lines from the individual whirls to be pinned down at the places where they pass through the axis of rotation of the system. In other words, the motions of the magnetic lines become calculated as though they partook of the rotational velocities of the frames of the whirls about their own axes, as well as of the rectilinear velocities of the whirls in their orbits.

(*F*) If the rotating system referred to in (*D*) is conducting instead of insulating; and, if, for the moment, we ignore the motional intensity represented by the last term of (11), it follows that the material will experience a charge distribution of such a nature as to completely annul, at all external points, the fields of the doublets referred to under (*C*), so that there will be no external field. If now we introduce the motional intensity, it will produce, in the conducting system, an additional distribution of electricity whose field inside the material just balances the motional intensity as represented by $[v \cdot B]/c$.

The field due to the charge distribution brought about by the motional intensity is the sole contribution to the electric field at external points. It further turns out that the magnitude of the external electric field is unaltered by the existence of a specific inductive capacity greater than unity in the material.

(*G*) The system referred to under (*F*) though conducting is nevertheless supposed to be insulated. The electrostatic distribution resulting from the motional intensity will be such as to cause a variation of electrostatic potential throughout the rotating body; and, the potential of the axis of rotation will not be zero. If this axis is subsequently earthed, an additional charge will come to the system. This charge will be distributed in the same way as it would be distributed on a non-magnetic body of the same size and shape; and, indeed, to a high order of approximation, its distribution will be the same as on the conductor at rest. Its total amount will be equal to minus the product of the capacity of the body and the potential of the axis before it was earthed. If some point other than the axis of rotation is earthed, the charge which comes to the body will be that necessary to bring this point to zero potential.

The case of practical importance is, of course, that in which some point of the system (usually the axis of rotation) is earthed; and, we thus see that, in this case, the external field is entirely determined by the electrostatic distributions resulting from (*a*) the direct action of the motional intensity as calculated from the induction as ordinarily defined for a point within the material, and (*b*), the charge which comes to the system as the result of a point on it being earthed.

(*H*) It is to be observed that the distribution resulting from (*a*) under (*G*) is of a type comprising a volume distribution and an equal surface distribution. These two distributions do not, in general, cancel each other's effects as regards the field at an external point. As regards alteration of *potential* of any fixed conductor which completely surrounds the rotating body, these two charges, owing to their equality, do cancel each other's influence, however; for the potential produced on a closed conductor by a point charge within it is independent of the location of the point charge. The charge resulting from (*b*) under (*G*) is thus the only agency active in causing alteration of potential of such an inclosing conductor.

(*I*) It is to be observed that all contributions to the field in the case of a symmetrical rotating system are of an electrostatic nature; and, there will consequently be no electric field in any space from which the magnetic system is completely shielded by an earthed conducting shield not participating in the rotation.

Discussion of Experimental Results Heretofore Obtained.

In line with what has been written above, we should expect that, in the case of a rotating magnetized system, there would be an external electric field of the type described in sections (*F*) and (*G*) of the summary

given above, and having its detailed origin in the considerations summarized in (*A*) to (*G*).

One of the first of the modern experiments on this matter was made by E. H. Kennard.[1] The essentials of this experiment are as follows:

A steel bar *B*, Fig. 3, was surrounded by a current-carrying solenoid *S*, in such a way that the bar could be rotated at high speed. The solenoid was fixed; it was surrounded by an earthed metal sheath *K*, and con-

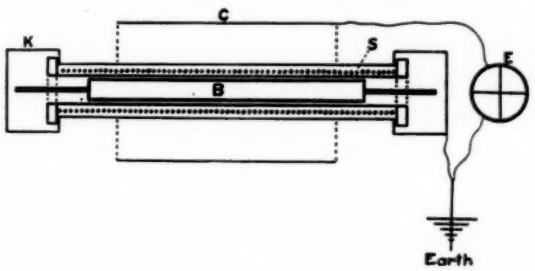

Fig. 3.

centric with this was an outer cylinder *C*. The cylinders *K* and *C* formed the two plates of a condenser, and were connected to the quadrants of an electrometer. No effect was obtained as a result of the rotation of the magnetic system.

It will be observed that the magnetic system was surrounded by a stationary earthed conductor, so that, in accordance with (*I*) in the summary above, no external electrical field could possibly arise since all possible electrical fields are of a type derivable from an electrostatic potential. Indeed, the earthed solenoid itself would have been sufficient to provide a shield even if the inner cylinder had been absent.

Apart from these considerations, however, even if the solenoid had rotated with the bar, and the shield *K* had been absent, there would have been no effect provided that the bar was insulated; for, the sole origin of a field in such a case would have to be sought in the surface and volume distributions of charge referred to under (*F*) above, and these, being equal and uniformly distributed as regards the cylindrical magnet, would practically cancel each other's effects at all points outside. The cancellation would be complete for an infinitely long bar. Even if the rotating body had not been in the form of a cylinder, however, there would have been no alteration in the potential of the outer member of the condenser, even though the earthed shields had been absent, provided that the rotating body itself was insulated. For, the total volume and surface charges being equal, there would, in accordance

[1] "Unipolar Induction," Phil. Mag., S. 6, 23, 937, 1912.

with (*H*) above, be no alteration of potential of any cylinder such as *C*, completely surrounding the rotating body.

If, however, the solenoid rotated with the bar, *K* were eliminated, and the axis of the rotating system were earthed, there would exist the field discussed under (*G*), and produced by the charge which comes to the system to maintain the potential of the axis zero.

In a later experiment by S. J. Barnett,[1] the two armatures of the condenser were surrounded by a coaxial solenoid which could be rotated about its axis. The outer cylinder was earthed, and the inner cylinder could be insulated or connected to the outer cylinder at will. It was found that rotation of the solenoid produced no effect. This is what we should expect in accordance with (*B*) and the arguments in the earlier portion of this paper, upon which this conclusion is based.

Barnett performed another experiment in which the magnetic field was produced by two electromagnets which were arranged so that they could be rotated about their axes of magnetization, which were in line. The condenser system was placed between the two electromagnets so that its axis was also the axis of the magnets. The ends of the outer cylinder were closed with brass caps so that the outer cylinder and these brass caps formed a complete shield around the inner cylinder, which could be insulated or earthed. In accordance therefore with (*I*), and quite apart from all other considerations, we should expect no alteration of potential of the inner cylinder as a result of the rotation of the magnets; and, no alteration of potential was obtained. In the absence of the brass caps, small effects should, theoretically, be obtained in accordance with the principles summarized under (*F*) and (*G*); but, they would have no simple relation to the effects as calculated by the ordinary application of the "Moving Line Theory."

It is perhaps worth while to emphasize the fact that we are not denying the possibility of an effect on the general grounds of supposing that a closed earthed shield will always prevent external actions from creating a field within. Such a conclusion would not be justified. The essentials of the argument are contained in the fact that such fields as are to be expected in experiments of this kind are fields determined by electrostatic potentials, in accordance with the views summarized in sections (*A*) to (*I*) above.

Barnett's experiment with a rotating solenoid, and without iron, was repeated by Kennard[2] with a modification which permitted of the rotation of the condenser system with the solenoid. A similar experiment

[1] PHYS. REV., 35, 323, 1912.

[2] Phil. Mag., 33, 179, 1917.

has also been performed by G. B. Pegram.[1] When the condenser system was stationary, no effect was of course observed. When the solenoid and condenser were rotated together, the potential difference set up between the condenser armatures, when connected together, was simply that calculated from the motional intensity produced in the connecting wire, as a result of its motion in the magnetic field. This potential difference was of course measured by an electrometer, the armatures being separated after the rotation was started, and the system being allowed to come to rest before readings were taken. Concerning this part of the effect there is presumably no divergence of opinion. It may be of interest to observe, however, that it would be obtained even though the wire connecting the armatures were outside the field. For, imagine the most general case as depicted in Fig. 4, where the connecting wire is

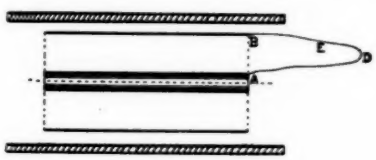

Fig. 4.

partly within the solenoid and partly outside. If H_n is the component of the field measured in the direction of the outwardly drawn normal to the surface (say the surface S) traced out by the wire $ADEB$, r the distance of an element of the wire from the axis of rotation, and ω the angular velocity, we have, for the potential difference between A and B as measured along this wire,

$$V = \frac{1}{c}\int (\omega r) H_n ds,$$

ds being an element of the path $ADEB$, and the integral being taken along the whole path. Now, $\int 2\pi r H_n ds$ is simply the total flux of magnetic force through the surface S, and is the same, therefore, as the magnetic flux through the annulus between A and B. Hence, if a and b are the radii of the two armatures, and H the magnetic field, which is uniform in the solenoid although it is not uniform all over the wire $ADEB$, we have:

$$V = \frac{H\omega}{c}\int_a^b r dr = \frac{H\omega}{2c}(b^2 - a^2),$$

which is the same as the value which would have been obtained if the connecting wire had gone straight from A to B. Uniformity of the field

[1] PHYS. REV., 10, 591, 1917.

over the path is not necessary; the only essential is that if there is any part of the wire which does not revolve, it shall be situated in a place where the magnetic field is negligibly small. We see that, even in the limiting case, where the wires pass out of the solenoid parallel to the axis, and are connected together at some remote point outside, the full value of V may be expected.

An experiment seeking to detect a field of electromagnetic induction due to the uniform *rectilinear* motion of a magnet has been recently performed by Barnett.[1]

It is obvious that, since the electric field of a simple amperian whirl moving with uniform rectilinear velocity obeys the "Moving Line Law," the resultant electric field of any magnet whose parts all move with the same uniform rectilinear velocity will be that calculated from the "Moving Line Law" as applied to the *resultant* magnetic field. The problem is thus much simpler than that of rotation. Moreover, it involves no extra field resulting from electrostatic distributions brought about in the magnet by the motional intensity; for, in the magnet itself, the forces on an element of charge, as a result of the moving line field, is just cancelled by the motional intensity, so that there is no electrostatic distribution due to this cause. If there are any fixed magnets in the system, they of course contribute nothing to the effect.

In Barnett's experiment, the attempt is made to charge a condenser by causing an electromagnet to swing past it, the electromagnet constituting the bob of a large pendulum. A fixed magnet also figured in the experiment for a reason which we need not enter into here; for, on the views expounded above, it produces no electrical effect whatever. No charging effect was found in the experiment.

The condenser and insulated system were shielded by an earthed case, to protect them from electrostatic effects. We shall show that the charge distribution set up in the case by the field of the moving lines produces, at all points within the case, a field just equal and opposite to the field which the moving lines produce there. No further comment is there necessary to show that no charging effect is to be expected in an experiment of this kind.

We cannot immediately import the ideas we have invoked in the case of uniform rotation to explain a shielding effect of the type referred to, because the vector potential is not independent of the time in the case of rectilinear motion; so that, it is not immediately obvious that the whole of the field E as given by (5) is derivable from an electrostatic potential.

[1] PHYS. REV., 12, 95, 1918.

In the direction parallel to the line of motion, the poles in Barnett's experiment were 33 cms. long. The corresponding dimension of the shielding case was 21 cms.; so that, since the field was sought at the instant during the swing of the magnet when the middle of the poles passed the condenser, and presumably therefore, when the 21 cms. of the case fell well within the poles, the problem may be treated as a two-dimensional one in the yz plane perpendicular to the line of motion of the magnet.

The "Moving Line Theory" gives, for the components E_y and E_z of the electric field, in terms of the components H_y and H_z of the magnetic field:

$$E_y = \frac{v}{c} H_z, \qquad E_z = -\frac{v}{c} H_y.$$

Now the magnetic field H is derivable from a potential Ω; and, as Ω is a solution of LaPlace's equation outside the magnet, we know, from the theory of functions of a complex variable, that it is always possible to find a potential Φ, itself a solution of LaPlace's equation in the region where Ω is a solution, and such that:

$$\frac{\partial \Phi}{\partial y} = \frac{v}{c} \cdot \frac{\partial \Omega}{\partial z}, \qquad \frac{\partial \Phi}{\partial z} = -\frac{v}{c} \cdot \frac{\partial \Omega}{\partial y}$$

Φ is, in fact, v/c times the function conjugate to Ω. Thus, we have:

$$E_y = -\frac{v}{c} \frac{\partial \Phi}{\partial y}, \qquad E_z = -\frac{v}{c} \frac{\partial \Phi}{\partial z},$$

showing that E is derivable from a potential, the apparent charges which give rise to it being situated *outside* the region in which Ω satisfies LaPlace's equation, and hence outside the shield, in the present instance. This is all we need in order to say that, under the influence of the potential $v\Phi/c$, the case will acquire a charge distribution such as to just annul the field which $v\Phi/c$ itself gives at all points within the case.[1]

An exactly analogous argument would show that, if the magnet were at rest, and the remainder of the apparatus were in uniform rectilinear motion, there would be complete shielding as regards the effect of the motional intensity on a charge which was within the shield, and moved in company with it.

A simpler, but interesting example of this type of action is afforded by the case of the motion of a closed conductor in a uniform magnetic field. The motional intensity acting on a charge within the conductor and moving with it would be uniform. In order to balance the effect

[1] This demonstration was presented by the writer at the meeting of the American Physical Society at St. Louis, December, 1919.

of this motional intensity on the surface, the conductor would experience a charge distribution; and, we know that the charge distribution which will produce, over the surface of the conductor, equilibrium under the action of the uniform field of force, will also produce complete cancellation of that field inside the conductor.

SOLUTION FOR A TYPICAL CASE OF UNIFORM ROTATION.

The rotation of a permanently and uniformly magnetized conducting sphere, whose axis of rotation is earthed, and whose direction of rotation is clockwise[1] *as viewed by an observer looking in the direction of the magnetization.*

The field will be represented entirely by the fields of the electrostatic distributions referred to under (*F*) and (*G*) in the summary given earlier in the paper. Let us first consider the results which would be obtained if the axis of rotation were insulated.

If r and θ are the polar coördinates of a point in the sphere, B the magnetic induction at a point inside the sphere, and ω the angular velocity, then, on account of the motional intensity, a unit of charge moving with the sphere will experience a force perpendicular to the axis of rotation; and, as shown in the earlier part of this paper, the magnitude of this force when measured outwardly from the axis of rotation is $B\omega r \operatorname{Sin}\theta/c$. Consequently, this represents the field which must be balanced by the electrostatic distribution. The electrostatic distribution is of the same type as that necessary to balance the effect of centrifugal force on the free electrons of a rotating sphere, since the magnitude of this force would be, for an electron of mass m, $mr\omega^2 \operatorname{Sin}\theta$. In a former paper,[2] I have given the solution of the latter problem; and, in order to adapt it to the present case, it is only necessary to replace $m\omega^2/e$ by $B\omega/c$, and then change the sign, since, in the paper referred to, the effect of centrifugal force on a *negative* electron was under consideration. Making this change, we find from equation (24) of the paper referred to, that the potential V is, for points outside the sphere,

$$V = -\frac{B\omega a^5}{3cr^3}\left(1 - \frac{3}{2}\cdot \operatorname{Sin}^2\theta\right). \tag{18}$$

At the surface of the sphere and on the equator, the electric field is

$$-\left(\frac{\partial V}{\partial r}\right)_a = \frac{1}{2}\cdot\frac{Bwa}{c}. \tag{19}$$

[1] And hence clockwise as viewed in the direction of the induction.

[2] Terr. Mag., 22, 162, 1917. The solution given in this paper is carried out in electromagnetic units; but, it is true as it stands when the quantities are in Heavisidean units.

If I is the intensity of magnetization, the demagnetizing field H_1 within the sphere is $I/3$, and the induction is $I - H_1$, *i.e.*, $2H_1$. Hence, since the tangential fields just inside and just outside of the surface are equal, the result given by (19) is the same as would be given by the ordinary application of the "Moving Line Theory." The forms of variation of the field over the surface of the sphere are quite different for the erroneous and correct method of calculation, however; and, at places where $\mathrm{Sin}^2\,\theta$ is less than $2/3$, even the signs are opposed. Equation (18) shows that the points on the sphere corresponding to the axis of rotation will have potential $-B\omega a^2/3c$. If the axis of rotation is now earthed, it will be necessary for the sphere to take a uniformly distributed charge q given by:

$$\frac{q}{4\pi a} = \frac{B\omega a^2}{3c}. \tag{20}$$

This will result in a contribution $B\omega a^3/3cr$ to the potential at external points, so that the complete expression for V is now:

$$V = -\frac{B\omega a^2}{3c}\left[\frac{a^3}{r^3}\left(1 - \frac{3}{2}\cdot \mathrm{Sin}^2\,\theta\right) - \frac{a}{r}\right]. \tag{21}$$

At a point on the surface of the sphere, the component of the field in the direction of the outwardly drawn normal is:

$$-\left(\frac{\partial V}{\partial r}\right)_a = -\frac{B\omega a}{c}\left(\frac{2}{3} - \frac{3}{2}\cdot \mathrm{Sin}^2\,\theta\right). \tag{22}$$

On the equator this is $5B\omega a/6c$, and is just $5/3$ times the field which we should have calculated from the "Moving Line Theory" as ordinarily applied, if we had ignored every effect other than the field due to the so-called motion of the lines of force. The ordinary application of the theory would not lead to the same type of distribution of field over the sphere as that given by (22).

In the case where the sphere is insulated, the total volume and surface distributions of charge are equal; so that, if we were to surround the sphere with a fixed insulating conducting sphere of radius b, there would be no change of potential of the outer sphere as a result of rotation of the inner sphere; although there would, of course, be rearrangement of charge over its surface. When the axis of rotation is earthed, however, the excess charge q, given by (20), which comes to the inner sphere, will produce a potential

$$W = \frac{B\omega a^3}{3cb} \tag{23}$$

in the outer sphere, since the potential of the latter is independent of

the distribution of charge within it. If W and B are in ordinary electrostatic and electromagnetic units respectively, the equation will remain unchanged. If the outer sphere were connected to an electrometer, we should have to replace b by the capacity of the whole insulated system, including the electrometer.

We have seen that the induction within a uniformly magnetized sphere is $2H_1$, where H_1 is the demagnetizing field within the sphere. It is, of course, impossible to magnetize a piece of steel to such an extent that the accompanying demagnetizing field which it produces within itself is greater than, or even as great as a fixed limit known as the "Coercivity" of the substance. Even for tungsten steel, the coercivity is only $52 \cdot 6$; so that 105 certainly represents a value of B greater than that actually attainable. If the inner sphere had a radius of 5 cms., and were rotated at a speed of 100 revolutions per second; and, if the outer sphere were connected to an electrometer such that the total capacity was 50 cms., we find by using (23), that the alteration of potential of the electrometer would certainly be less than 18×10^{-7} E.S.U., *i.e.*, $5 \cdot 4 \times 10^{-4}$ volt.

The calculation for the case of an iron sphere inductively magnetized by an external magnetic field is the same as that for the permanently magnetized sphere cited above. B represents the resultant induction within the sphere, and is related to the external field H by the expression:[1]

$$B = \frac{3\mu}{\mu + 2} \cdot H,$$

so that if μ is of the order 2000 C.G.S. unit, as in the case of soft iron, B is practically three times the external field. For a case where H is 2,000 C.G.S. units, the alteration in potential of the insulated sphere system cited in connection with the permanently magnetized sphere will be 3×10^{-2} volt. If the sphere were of non-magnetic material, such as copper, the alteration in potential would be one third of this amount. The influence of permeability would, of course, be greater in the case of elongated bodies such as ellipsoids.

No special interest would attach to measurements made upon a copper sphere, for, presumably, nobody would doubt that, under the influence of the motional intensity, a copper sphere would experience the sort of charge distribution, etc., we have discussed. Perhaps a greater interest lies in the value for the iron sphere rotating in an external magnetic field; for we may here look upon the effect as composed of two parts, the part corresponding to the rotation in the external field, which part is the same as for a copper sphere, and the part due to the rotation in the

[1] See for example J. J. Thomson's Elements of Electricity and Magnetism, p. 260.

field of induction resulting from the magnetization of the sphere itself. The existence of the latter part as would be shown by the increase of the effect on subsituting an iron sphere for a copper sphere, is perhaps of some interest, as it arises from the magnetic induction of a rotating body; and, to one who thinks in terms of moving magnetic fields, there might be some doubts concerning it, although, from the standpoint of the development given in this paper, the reality of its existence seems perfectly clear.

THE ROTATION OF A MAGNETIC DOUBLET ABOUT AN AXIS PASSING THROUGH ITS CENTER AND PERPENDICULAR TO ITS OWN AXIS.[1]

While this problem is not primarily involved in the discussion of the usual experiments on electromagnetic induction and relative motion, it is of considerable interest. With the doublet considered as the representative of an amperian whirl, the true solution for the field is, of course, that corresponding to equation (5).

Without some theory for rotation, analogous to the theory of relativity for rectilinear motion, we are unable to say what would happen to the constitution of the amperian whirl as a result of the rotation; we are unable to predict what ψ would be. If, however, we assume that the constitution of the whirl is not altered by the rotation, the field will be given by the first term of the right-hand side of (5).

The vector potential of a magnetic doublet of moment μ_x, with its axis parallel to the axis of x is given by:[2]

$$U_x = 0, \qquad U_y = \mu_x \frac{\partial}{\partial z}\left(\frac{1}{r}\right), \qquad U_z = -\mu_x \frac{\partial}{\partial y}\left(\frac{1}{r}\right)$$

with corresponding expressions for the doublets parallel to the other axes.

Without sacrifice of generality, we may take the y axis as the axis of rotation, and the z axis as the axis of the doublet at some particular instant. If we assume that the magnetic field of the doublet is, at each instant, the same as if the doublet were at rest, we shall merely neglect effects of a small order of magnitude; and, in this case, the change of moment per second corresponds to the creation of moment along the x axis at the rate $m_0 h\omega$, that is $\mu\omega$ units per second, h being the length of the doublet, m_0 its magnetic pole strength, and μ its magnetic moment. Hence, since the components of the electric field are equal to minus $1/c$ times the rates of increase of the corresponding components of the vector potential, we have for the field components:

$$E_x = 0, \qquad E_y = -\frac{\omega\mu}{c} \cdot \frac{\partial}{\partial z}\left(\frac{1}{r}\right), \qquad E_z = \frac{\omega\mu}{c} \cdot \frac{\partial}{\partial y}\left(\frac{1}{r}\right). \qquad (24)$$

[1] A paper on this section was presented by the writer at a meeting of the American Physical Society at Chicago, November 29, 1919.

[2] See, for example, J. H. Jeans's Electricity and Magnetism, page 394.

Now suppose we calculate the electric field which would be given by the application of the "Moving Line Theory" to the problem on the understanding that the magnetic lines from each pole are to be treated separately, and are to partake of the rectilinear velocity of the pole in its orbit, but not of any rotary motion. At the instant when the axis of the doublet is parallel to the z axis, the "Moving Line Theory" as applied to one of the poles gives 0, $\omega h H_z/2c$, $-\omega h H_y/2c$ for the components of the electric field, $\omega h/2$ being the velocity of the pole, and H_y and H_z being the y and z magnetic field components at the point at which the electric field is sought, in so far as they are determined by the single pole under consideration. Treating the effects of the two poles as additive as regards the electric fields, putting l, m, n for the directional cosines of the radius vector, and observing that, for a single pole,

$$H_y = \frac{m_0 m}{r^2}, \qquad H_z = \frac{m_0 n}{r^2}$$

we have, from the "Moving Line Theory" in this form:

$$E_x = 0, \qquad E_y = \frac{\omega h}{c}\left(\frac{m_0 n}{r^2}\right), \qquad E_z = -\frac{\omega h}{c}\left(\frac{m_0 m}{r^2}\right),$$

i.e.,

$$E_x = 0, \qquad E_y = -\frac{\omega \mu}{c} \cdot \frac{\partial}{\partial z}\left(\frac{1}{r}\right), \qquad E_z = \frac{\omega \mu}{c} \cdot \frac{\partial}{\partial y}\left(\frac{1}{r}\right),$$

which agree with the results obtained in (24) from the Lorentzian theory. Although we have utilized the idea of a pole-distance in the above calculation, the results naturally do not involve this pole-distance explicitly, but depend only upon the moment of the doublet.

We thus see that the ideas controlling the application of the "Moving Line Theory" to this case are an extension of, and bear a close analogy to those applying to the case where the doublet revolves about some center outside itself. If, for the moment, we utilize Fig. 2 for a purpose different from that for which it functioned before, and suppose N to represent one of the magnetic poles of the doublet, and O the center of the doublet, we see that, on applying the "Moving Line Theory" to this case, on passing from the position N to the position N_1, we must calculate the velocity of a magnetic line NP as though it had assumed the orientation N_1P in diagram B, and not as though it had assumed the orientation N_1P in diagram C. In fact, while we must imagine the magnetic lines to be rigidly attached to the pole, we must suppose that the latter does not rotate about an axis within itself as it revolves about the center of the doublet. Any calculation founded upon the idea of the resultant lines of force of the doublet turning with the doublet as though rigidly

attached to it would be the equivalent of supposing the lines due to the individual poles to partake of the rotary motion referred to above, as well as of the rectilinear motion. Such a view would lead to absurdities from the outset; for, it would obviously predict infinite velocities for the lines at infinite distances from the doublet. It would be analogous to what we should be driven to if, in the case of an electron revolving in an orbit, we were to assume the tubes of electric force to be rigidly attached to the electron, while the latter partook of the kind of motion it would experience if stuck on to a spoke emanating from the center of the orbit. In this case, even the importation of the ideas of finite rate of propagation of effects could do no better than leave us with the picture of the tubes of force winding themselves up into spirals as the electron continued to describe its orbit over and over again. A proper calculation of the effects due to a revolving electron simply represents the tubes as moving backwards and forwards as the electron passes from one extremity of a diameter to the other; and, it is never necessary to suppose the tubes to have a velocity greater than that of the electron itself. A similar remark applies in the case of the rotating magnetic doublet.

It may be of interest to remark that, if we calculate the electric field due to a rotating doublet, at a point on a prolongation of its axis, the erroneous method of calculation, which treats the resultant magnetic lines as moving rigidly with the doublet, gives just twice the value resulting from the correct method of calculation.

In conclusion it may be observed that, to the extent that the magnetic field of a moving amperian whirl may be considered as equivalent to that of a magnetic doublet, the electrical field resulting from the most general type of motion of the whirl, and hence of a non-conducting magnet (of unit specific inductive capacity) in bulk, may be calculated from the "Moving Line Theory" as applied individually to the magnetic lines from the separate representative magnetic poles, provided that the velocities of the lines are calculated from the rectilinear velocities of the poles, and not in a manner such as to make them partake of any rotary velocity of the poles about their own centers. In the case of a conducting magnet, the portion of the above field derivable from a potential will become cancelled by the distribution of charge which it sets up in the magnet itself, as stated under (*F*) of the summary given earlier in the paper. We shall be left with the part $-\frac{1}{c}\cdot\frac{\partial U}{\partial t}$; and, superposed upon this, we shall have the fields of the electrostatic distributions produced by this part, and by the motional intensity resulting from the motion of the magnet in its own magnetic field, and discussed under (*F*), (*G*) and (*H*) of the summary above referred to.

The ideas developed in this paper arose during the process of writing up an account of some experimental work which will be published later, and for which funds were provided by the executive committee of the graduate school of the University of Minnesota. The writer desires to take this opportunity of expressing his thanks to the committee.

DEPARTMENT OF PHYSICS,
UNIVERSITY OF MINNESOTA,
December 27, 1919.

THE PRESSURE OF SOUND.

BY WARREN WEAVER.

SYNOPSIS.

The Pressure of Sound: Relation between Pressure and Energy Density.—An argument is given, following a method used by Larmor, to show that a certain general type of radiation will exert a pressure. The pressure of small sound waves is found to agree with this result, but for finite waves the conditions for the application of the argument are not satisfied. These finite waves do exert a pressure which depends upon the relation between pressure and density, the pressure being zero in a certain important case. This theory has been developed by Lord Rayleigh. It however appears that any actual aërial wave does exert a pressure not zero. The pressure on an absorbing sphere is a second order effect in the product (ak), and is therefore not considered in the usual treatment of spherical obstacles. Waves of energy density of 0.5 ergs/cm.^{3} or greater apparently must be treated as finite.

1. It is interesting and surprising that the subject of the steady pressure of sound waves on a surface normal to the direction of propagation has been so little mentioned in the ordinary literature of the subject. One finds, for example, no mention of it in Rayleigh's treatise, in Lamb's Dynamical Theory of Sound, and in many standard texts on physics. It is treated in two articles by Rayleigh,[1] in one by Poynting,[2] and in the article on sound by Stokes in the Encyclopedia Britannica. There is an apparent but entirely superficial confusion in the treatments here cited which it is the purpose of this note to remove.

2. To Larmor is due a method of argument to show that any propagated disturbance in which the energy density in the beam is inversely proportional to the square of the wave-length will exert a radiation pressure. For let the disturbance be propagated with a velocity c, and let it be reflected by a plane normal to the direction of propagation moving with a velocity of magnitude v opposite to c. Then by Doppler's Principle the wave-length of the reflected beam will be reduced in the ratio $1 - 2v/c$ to 1, so that the energy density in the reflected beam will be increased in the ratio $1 + 4v/c$ to 1, (v/c being supposed small). Let e be the energy density in the original beam, and consider unit area of the reflecting surface. An amount of energy $e(c + v)$ will be encountered by it per second. It will be reflected in a wave train that is shorter than $(c + v)$ in the ratio $1 - 2v/c$ to 1, but in which the energy density is

[1] 1905, II., p. 364; 1902, I., p. 338, Phil. Mag.

[2] 1905, I., p. 393, Phil. Mag.

larger in the ratio $1 + 4v/c$ to 1. Accordingly there will be added, per second, energy equal to

$$e(c + v)(1 - 2v/c)(1 + 4v/c) - e(c + v) = e(c + v)2v/c.$$

This energy is supplied by work done in advancing the reflecting surface a distance v per second against a pressure p. So that:

$$pv = e(c + v)2v/c$$

or

$$p = e(c + v)2/c.$$

The total energy density is the energy density in the oncoming and in the reflected beams, so that, if we denote it by E,

$$E = e + e(1 + 4v/c) = 2e(1 + 2v/c),$$

and if we set

$$p = \kappa E$$

we have, neglecting first order terms in v/c,

$$\kappa = 1$$

or

$$p = E, \tag{1}$$

so that the pressure is equal to the total energy density in the wave train if the reflector is moving slowly as compared to the velocity of propagation of the disturbance. In case the reflector is not moving at all, so that $v = 0$ the result (1) is rigidly correct. This argument of course covers the case of light pressure on a normal reflecting surface.

The argument, as given by Larmor,[1] is not restricted to first order terms in v/c. An incident train of length $c + v$ is reflected into a train of length $c - v$. The energy density in the reflected train is accordingly

$$e\left(\frac{c + v}{c - v}\right)^2$$

and the total energy reflected per second

$$e\left(\frac{c + v}{c - v}\right)^2 (c - v).$$

We have then for the increase in energy per second

$$e\left(\frac{c + v}{c - v}\right)^2 (c - v) - e(c + v) = e(c + v)\left[\frac{c + v}{c - v} - 1\right] = pv.$$

The total energy density in front of the reflector is

$$E = e + e\left(\frac{c + v}{c - v}\right)^2$$

[1] Art. on Radiation, Enc. Brit., 11th ed.

and if we set as before

$$p = \kappa E$$

or

$$\frac{e(c+v)}{v}\left[\frac{c+v}{c-v} - 1\right] = \kappa e\left[1 + \left(\frac{c+v}{c-v}\right)^2\right]$$

we have

$$\kappa = \frac{c^2 - v^2}{c^2 + v^2}, \tag{2}$$

which reduces to unity as in (1) if we neglect second order terms in v/c.

3. Let us consider a simple harmonic train of waves travelling in the positive x-direction given by

$$\varphi = Ae^{i(nt-kx)}. \tag{3}$$

The energy per unit volume of this plane wave disturbance is inversely proportional to the square of the wave-length, and hence sound of this sort should cause a pressure upon a reflecting surface equal to the total energy density in the incident and reflected sound beams according to 2. We shall see, in fact, from elementary mechanical principles that in the case of a perfectly absorbing plane obstacle normal to x there is also a pressure equal to the total energy density in front of the obstacle. Let R be the absorbing surface, and let a, b, c, d be a column one square centimeter in cross section, of any length, and normal to R. Since in

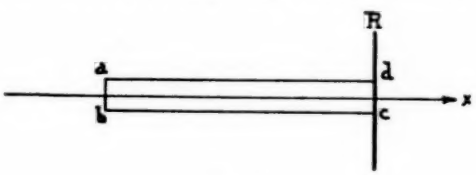

Fig. 1.

a steady state the air within the column neither gains nor loses momentum the momentum flow across a–b per second will be equal to the pressure on R. On the average there will be no momentum flow due to the variable part of the pressure, that is due to $\rho(\partial\varphi/\partial t)$, since this obviously has a time average of zero. The steady pressure at a–b will of course cause a steady pressure on R, but with that we are not concerned. The volume of air, however, gains forward momentum by having air enter it moving to the right, and as well by having air leave it moving to the left. If the velocity of the air particles be u the instantaneous rate at which momentum is flowing to the right across the surface a–b is then ρu^2: so that

$$p = \rho u^2 = \rho\left[\frac{\partial\varphi}{\partial x}\right]^2.$$

But from (3) we have

$$\frac{\partial \varphi}{\partial x} = -ikAe^{i(nt-kx)},$$

the real part of which

$$= kA \sin (nt - kx).$$

We have, therefore,

$$p = \rho k^2 A^2 \sin^2 (nt - kx),$$

the average value of which over a whole number of periods

$$= \tfrac{1}{2}\rho k^2 A^2, \tag{4}$$

the known expression for the energy density in the incident beam. Equation (4) also gives, as is well known, the value approached by the average over any lapse of time as the interval becomes long with respect to the period.

4. Expression (3) above is the velocity potential for plane waves under the assumption of small particle velocities, or, what comes to the same thing, under assumption that the pressure variation is so small that the volume modulus of elasticity of the gas may be considered constant over this pressure range. The relation between pressure and density in an actual gas is however such that a wave cannot be propagated without change of type (Rayleigh, Theory of Sound, Art. 250). An approximate study of this change of type shows that the pressure crests travel with higher velocity than do the pressure troughs, so that there is a tendency for the wave to "comb over" as a water wave does near the beach. This implies that the forward flow of momentum across a surface will be larger than in the case just considered, since the more dense portions of the gas are moving forward the more rapidly. There is actually a resultant forward flow of matter, which a reflecting or absorbing surface would have to reverse or annihilate, so that we should expect the pressure upon it to be greater than in the case given by (3). This problem has been treated by Rayleigh and Lamb. Rayleigh abandons the ordinary sound equations and starts from the basis of Bernoulli's equation. Lamb retains the ordinary equations but finds a corrective term to the expression for the change in pressure due to a small change in volume, the correction being obtained from the ordinary gas law. The result obtained is that the pressure is given by the equation

$$p = \tfrac{1}{2}(\gamma + 1)(\text{average energy/cubic centimeter}). \tag{5}$$

This is, in the first place, a hybrid result. As actually obtained the pressure is given by

$$p = \tfrac{1}{2}(\gamma + 1)\frac{1}{c}\int_0^{+c} \rho_0 u^2 dx.$$

The expression

$$\frac{1}{c}\int_0^c \rho_0 u^2 dx,$$

is the value of the average energy density under the assumption of small pressure changes, while the coefficient $\frac{1}{2}(\gamma + 1)$ differs from unity only because the pressure changes have not been assumed to be so small. The result is therefore not to be considered as establishing an exception to the principle given in 2. In fact in this case we could not apply the principle given by Larmor at all because, there being continual change of type, there is, strictly speaking, nothing one can call the wave-length at all.

5. If the changes of pressure are small enough a wave can be propagated without change of type whatever the law between pressure and density. This is equivalent to saying that whatever the relation between p and ρ a sufficiently short piece of the p, v curve may be considered a straight line, such a relation being that which makes possible propagation without change of type.[1] Within the range over which this approximation is allowable the pressure will be numerically equal to the energy density in the sound-filled space before the reflector or absorber. In the case of finite waves, as suggested above, the pressure is in general larger than this. The exact relationship depends upon the law connecting pressure with density. It is given by equation (5) when the law of pressure is that given by the adiabatic relationship

$$p/p_0 = (\rho/\rho_0)\gamma.$$

(5) reduces to (1) in case we have Boyle's law. For the case of the law

$$p = \text{const} - d^2\rho_0^2/\rho, \qquad (6)$$

which is the only relation under which there can be propagation without change of type, the pressure comes out curiously enough to be zero. Lord Rayleigh therefore remarks that "pressure and momentum are here associated with the tendency of waves to alter their forms as they proceed on their course." This might seem to imply that waves whose type is preserved as they move do not exert a pressure and have no momentum associated with them. In the case of actual waves, however, equation (6) holds only over pressure ranges so small that (3) accurately gives the velocity potential, and (4) gives the pressure. For these very small waves Lord Rayleigh has found a pressure equivalent to that given by (4).[2] The conclusion is, then, that any actual aerial waves, whether

[1] Lamb, Dynamical The. of Sound, p. 175.

[2] Phil. Mag., 1902, p. 338.

of such magnitude as to be considered small or finite, whether their type is preserved or not, do exert a pressure.

Both equations (1) and (5) have been made the basis of experimental determinations of the energy of sound waves. W. Zernov[1] used powerful waves of frequency 512 and energy density of the order of 0.5 ergs/cm³., and found that equation (5) gave results which checked with a maximum discrepancy of 3 per cent. those given by a vibration-manometer method as developed by Wien.[2] W. Altberg's measurements[3] were on sound waves whose energy content was about half the above value, and he used equation 1. (His experiments were made before the publishing of Lord Rayleigh's 1905 paper.) It is unfortunately not possible to deduce from his results whether (1) or (5) represents the more closely the truth for sounds of this intensity since he considered only the constancy of the ratio of the result obtained by the pressure method to that given by the vibration-manometer method. This ratio was found to be approximately constant, as would of course be the case whether (1) or (5) was used.

6. The ordinary theory of the impinging of plane waves of sound on an obstructing sphere is upon the basis that (ka) is small, where a is the radius of the sphere. Since it exerts a pressure, we may associate a momentum with a sound beam, and since a perfectly absorbing sphere would annihilate per second the sound contained in a cylinder c in length and πa^2 in cross-section, it would be subjected to a pressure equal to $\frac{1}{2}\pi\rho A^2(ka)^2$. It is thus clear that for obstacles small enough to have the ordinary theory apply to them with good approximation the pressure effect we are considering would be negligible, containing as it does the square of (ka). This explains the absence of reference to any such pressure in the ordinary treatments. For sound in air of frequency 1,000 per second the product (ka) is equal to 0.01 (whose square we might perhaps agree to neglect) when $a = 0.0525$. Obviously, however, a criterion for how small particles could be and have the steady pressure effect sensible would have to take account also of density and amplitude. In certain experiments carried out during the war use was made of a small absorbing cylinder to measure pressure and hence energy density of super-sonic waves in water. The wave-lengths used were about those of the upper limit of audibility in air, their frequency being about four times this limit. The product (ka) in these experiments was approximately unity, so that the ordinary theory is entirely inapplicable, while the pressure is sensible and can be easily measured.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA.

[1] Ann. d. Phy., 21, 1906, p. 131.

[2] Wied. Ann., 36, 1889, p. 835.

[3] Ann. d. Phy., 11, 1903, p. 405.

THE MEASUREMENT OF RECEIVED RADIO CURRENTS WITH THE ELECTROMETER.

By E. O. Hulburt and G. Breit.

Synopsis.

Measurement of Received Radio Currents.—A new method has been developed for the accurate measurement of received currents in radio telegraphy by means of the *quadrant electrometer connected in the plate circuit of an electron tube* which is used either singly or as the detector tube of a multistage amplifier.

Calibration of the Apparatus.—In order to employ such an arrangement to measure signal intensity the relation between electrometer deflection and effective input voltage must be known, and a *condenser potentiometer* for such a calibration has been devised.

THE problem of measuring accurately the comparatively small currents produced in a receiving apparatus by waves transmitted from a distant radio station has not received up to the present time a completely satisfactory solution. Methods which have been used, such as the crystal detector and galvanometer, the shunted telephone receiver, etc., have been open to objection on the grounds of unreliability, unknown effect upon the receiving circuit, etc. A sensitive and dependable method would be particularly useful in studies of the propagation of electromagnetic waves and in studies of the characteristics of transmission and reception apparatus. In the present paper is described a method for the measurements of the received currents in radiotelegraphy which is believed to be an improvement over the methods used heretofore. The method involves the use of the quadrant electrometer and has been found to be workable and sensitive.

The electrical connections of the apparatus are shown diagrammatically in Fig. 1. The Dolezalek quadrant electrometer A is connected across a resistance R placed in the plate circuit of a three electrode electron tube T. This electron tube may be used alone, or it may be employed as the detector tube of a multistage amplifier. k is the contact key for short-circuiting the electrometer quadrants; n is the electrometer needle, which is charged to a low (or high) potential. The difference of potential of the electrometer quadrants due to the plate current flowing through the resistance R is compensated by a voltage divider M, Fig. 1. Adjustment is so made that the electrometer needle rests approximately at zero when no signals are being received. Then

when a signal is received the *mean value* of the plate current is altered on account of the rectifying action of the tube, and a deflection of the electrometer needle results, which is directly proportional to the change in the *mean value* of the plate current.

The method just outlined is not restricted to the use of the quadrant electrometer, but may with suitable modification be adapted to other types of current or voltage measuring instruments. The electrometer,

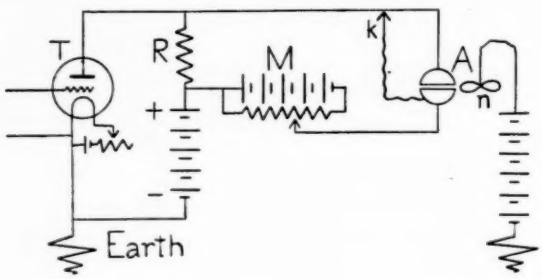

Fig. 1.

however, has an advantage over an instrument such as the galvanometer in experiments which require a knowledge of the electrical constants of the circuit, because the addition of the electrometer does not greatly complicate their determination.

Tests were made with a single tube, with a two and three stage high frequency resistance-capacity-coupled amplifier, and with a two and three stage high frequency transformer-coupled amplifier. When the electrometer needle was charged to 400 volts, its period being 15 seconds, the sensibility of the electrometer was 2,500 mm. deflection per volt difference of potential of the quadrants at a scale distance of four meters. The resistance R consisted of a noninductive resistance of 20 to 60 thousand ohms, in series with the telephone receivers. Thus, damped wave signals could be heard and the needle deflection observed simultaneously. With the above electrometer sensibility large deflections, *i.e.*, over 50 cms., were produced by spark signals of medium intensity in the telephones. Continuous wave signals also produced deflections, though of course these were not heard in the telephones. By using a separate heterodyne the continuous wave signals could be heard at the same time that the deflections were observed. Tuning the plate circuit of the first tube, so as to receive continuous waves by self-heterodyne, resulted in unsteadiness in the receiving system, making it difficult to obtain reliable deflections. It was found that, although the ordinary precautions had to be observed in the arrangement of the apparatus,

no elaborate shielding was necessary. A solid table on which to mount the electrometer and good storage cells sufficed to afford the system a satisfactory mechanical and electrical steadiness.

In view of the fact that each amplifier, and perhaps each tube, possesses amplifying and rectifying characteristics which depend upon the frequency, amplitude, wave-form, etc., of the impressed voltage, the amplifier must be calibrated to determine the relation between the voltage impressed on the grid of the first tube of the amplifier and the deflection of the electrometer. This calibration was carried out by the use of what may be termed a condenser potentiometer. The arrangement of the apparatus is shown in Fig. 2. Three variable condensers of capaci-

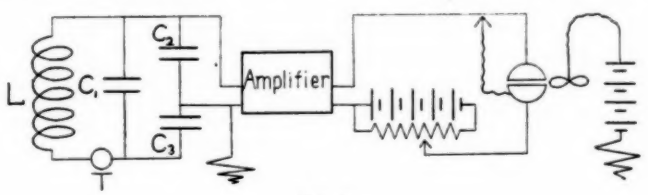

Fig. 2.

ties c_1, c_2, and c_3, are connected with a coil L in which high frequency current is induced by means of a suitable generating set. A thermo-galvanometer T measures the total current through the combination of condensers. Let the effective value of the current through T be I, let the frequency of the current be $\omega/2\pi$, and let the effective value of the voltage across c_2 be v_2. Then it may be shown that

$$v_2 = \frac{c_3 I}{\omega(c_1c_2 + c_1c_3 + c_2c_3)}.$$

By adjusting c_3 to a small value and c_2 to a large value, the coefficient of I in the above equation may be made so small that a readable value of I is obtained when v_2 is of the order of magnitude of the voltage produced by the current from radio signals.

After the calibration, *i.e.*, the relation between the impressed voltage and electrometer deflection, has been effected, the condenser c_2 may be replaced by the condenser of an antenna receiving circuit and the apparatus may then be employed to study signal intensity. Further, calibration curves of an amplifier for waves of various frequencies, types of modulation, etc., yield information concerning the amplifying and rectifying characteristics of the amplifier.

It is of essential importance to note that the use of the amplifier electrometer arrangement to measure signal intensity requires calibration with a wave identical in all respects with the wave of the received signal.

To state this in more detail:—in order to measure the intensity of an undamped sine wave, a calibration with an undamped sine wave must be carried out; the measurement of a modulated sine wave requires calibration with an identically modulated sine wave; similarly for spark signals; etc.

Two three-stage high frequency amplifiers, one resistance-capacity-coupled, the other transformer-coupled, were calibrated by the method described above for undamped sine waves. It was found necessary, in order to eliminate direct action between the generating set and the amplifier, to use twisted leads about five meters in length between L and c_1, and between c_2 and the amplifier, Fig. 2. Tests showed that the three condensers c_1, c_2, c_3, Fig. 2, could be placed in close proximity to each other unshielded, without producing disturbing mutual capacity effects. Measurements of the two amplifiers showed that the electrometer deflection was approximately proportional to the square of the mean value of the impressed voltage throughout the range of voltages used. This means that the rectifying action of the last electron tube is represented approximately by the Taylor expansion of the plate-current grid-voltage relation in which derivatives of higher order than the second are neglected. In Table I. is given a typical set of calibration measure-

TABLE I.

$V_{\frac{1}{2}}^2$.	Deflection.	$\frac{\text{Deflection}}{V_{\frac{1}{2}}^2}$.
0.00442 volts^2.....	38.6 cms.	8.73×10^3
0.00292	22.6	7.74
0.00229	17.0	7.42
0.00141	10.1	7.17
0.000734	5.4	7.36

ments of a three-stage high frequency transformer-coupled amplifier for an undamped sine wave of wave-length 850 meters. It was found that the calibration did not change appreciably when the values of the filament currents and plate voltages were purposely varied by four per cent. Thus when the calibration was once effected, reliable measurements of the intensity of received signals could be made with no more than the usual attention to current and voltage regulation of the amplifier tubes. A single calibration remained useful for several weeks, subsequent ones agreeing with the first, showing that no important changes had taken place in the operating characteristics of the amplifier.

JOHNS HOPKINS UNIVERSITY.
December, 1919.

ON THE INPUT IMPEDANCE OF THE THERMIONIC AMPLIFIER.

By Stuart Ballantine.

Synopsis.

At low frequencies the input circuit of the three electrode vacuum tube behaves as a pure capacity. At radio frequencies, due to the inherent capacity of the grid and plate electrodes, the *input impedance resembles the combination of a condenser shunted by a resistance or conductance.* The effective capacity and effective conductance are then functions of the load in the plate circuit. In the case of a resistance connected in this circuit, the conductance is positive and represents a loss. An *inductance load manifests itself in a negative conductance* and thus supplies energy from the plate to grid circuit. These *feed-back effects* are *occasioned solely by the grid-plate capacity* and depend upon its magnitude.

The phenomena described are investigated analytically and expressions are developed for the values of the effective capacity and conductance for various loading conditions. The physical significance of the equations is depicted by a series of curves, plotted for the various load configurations of the analysis. One set of curves shows the conditions which favor feed-back action and suggest the *principles which should govern the design of amplifying tubes.* Equations are developed which are of practical value in *choosing proper inductances for regenerative receiving circuits.* The desirability of so designing tubes as to *avoid excessive grid-plate capacity* is emphasized. Other curves show the variation of input inductance and capacity with pure resistance load. The reason for the failure of many radio frequency cascade amplifier arrangements is at once evident from the conclusions.

IT has been known for some time, at least since the publication[1] of Mr. Armstrong's important researches on the regenerative influence of an inductance in the plate circuit upon the current in the supply network, that the three electrode vacuum tube or thermionic amplifier, is not strictly a unilateral potential operated device but that a definite amount of energy is transferred from the plate circuit to the grid circuit at radio frequencies through the inter-electrodic capacities of the tube itself. Experience gained in the experimental study of cascade amplifier systems operating at radio frequencies has served to substantiate this view and has indicated the importance of suitably extending the circuit equations which depict the performance of the device to include this capacity between the input and output circuits.

By purely physical reasoning it is conceivable that when the plate circuit of the tube is loaded, either with inductance or resistance or both,

[1] Armstrong, Proceedings of the Institute of Radio Engineers, Vol. 3, No. 3, p. 215 (1915).

as the grid potential becomes *positive* the plate potential becomes *negative* by an amount which increases with the load impedance, the difference in potential thus generated will serve to force a current through the grid-plate capacity. Such a current will tend to increase the grid potential or to decrease it (provided there is an impedance in the supply circuit) depending upon its magnitude and phase. This effect has been noted by a number of observers working with the vacuum tube at radio frequencies and has been treated in its audio frequency aspect in a paper (unpublished) by Professor Morecroft. It is proposed in this paper to study in a formal analysis the radio frequency situation and to derive general expressions for the apparent impedance of the input circuit of the tube.

Fig. 1 depicts the elementary tube circuit, the input characteristics of which are to be investigated. It is understood that the various

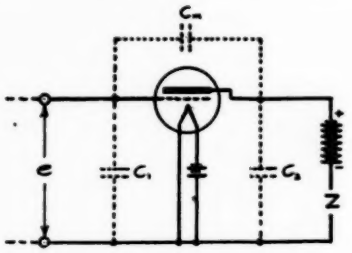

Fig. 1.

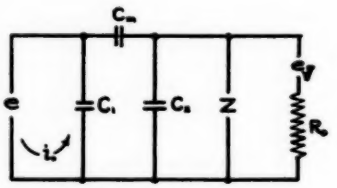

Fig. 2.

capacities denoted by C_1, C_2 and C_m represent in addition to the capacities of the tube elements, that of the connected apparatus, including for example, the leads to the base, the tube receptacle and the connecting wires to the external circuit. C_m represents the grid-plate capacity, which may be regarded in a certain sense as a *mutual* capacity and connecting link between the input and output circuits. This turns out to be the most important factor in the whole investigation and it is upon this that the feed-back phenomenon depends. Fig. 2 is the analytical equivalent of Fig. 1. Here e_p is the effective plate voltage equal to $\mu_0 e$ and e is the voltage applied to the network. The plate circuit load is designated by Z and may be of any form. The *input impedance* is defined as the ratio of the impressed force e to the current i_0 in the main mesh, the value of which may be conveniently determined by superposing the effects of e and e_p. Let the current due to e be denoted by i and that due to e_p by i'. The procedure consists in finding $i + i'$.

The sinusoidal voltage e encounters the symbolic impedance:

$$Z_{0,0} = \frac{\frac{1}{j\omega C_1}\left(\frac{1}{j\omega C_m} + \frac{R_0 Z}{R_0 + Z + j\omega C_2 R_0 Z}\right)}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_m} + \frac{R_0 Z}{R_0 + Z + j\omega C_2 R_0 Z}}, \tag{3}$$

giving for the current

$$i = \frac{j\omega\{C_m[R_0 + Z + j\omega C_2 R_0 Z] + C_1[R_0 + Z + j\omega R_0 Z(C_2 + C_m)]\}e}{R_0 + Z + j\omega R_0 Z(C_2 + C_m)} \tag{4}$$

due to the activity of e, e_p being temporarily disregarded. The feed-back current is similarly easy to find. What is required is the current in mesh C_m due to e_p (Fig. 3). C_1 does not enter the equations because it is short circuited by the assumed zero impedance of the supply circuit. The appropriate impedance operator is:

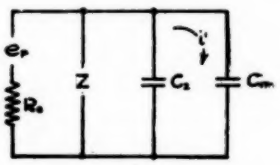

Fig. 3.

$$Z_{0,m} = \frac{\frac{Z}{j\omega C_2}}{R_0\left(\frac{Z}{j\omega C_2} + \frac{Z}{j\omega C_m} + \frac{1}{j\omega C_m j\omega C_2}\right) + \frac{Z}{j\omega C_m j\omega C_2}}, \tag{5}$$

yielding

$$i' = \frac{j\omega C_m Z}{R_0 + Z + j\omega Z R_0(C_2 + C_m)} e_p \tag{6}$$

for the feed-back current. Substituting $e_p = \mu_0 e$ (where μ_0 is that of the tube alone) and adding currents we have for i_0:

$$i_0 = i + i' = \frac{j\omega\{Z[C_1 + C_m(1 + \mu_0) + j\omega R_0(C_1C_m + C_2C_m + C_1C_2)] + R_0[C_1 + C_m]\}e}{R_0 + Z[1 + j\omega R_0(C_2 + C_m)]}. \tag{7}$$

This is of the general form:

$$i = \frac{a + jb}{c + jd} e. \tag{8}$$

It is proposed to resolve this current into two components, a real part which represents the current through the shunt conductance and an imaginary part which may be identified with the displacement current through the condenser. The current through such a combination of capacity and shunted conductance is given by:

$$i = (g + j\omega C)e, \tag{9}$$

where C and g represent the capacity and conductance respectively. The rationalization of the denominator is effected in the usual way and yields:

$$i = \frac{(a + jb)(c - jd)}{c^2 + d^2} e = \frac{(ac + bd) + j(cb - ad)}{c^2 + d^2} e. \tag{10}$$

Taking the real part

$$g = \frac{ac + bd}{c^2 + d^2} \tag{11}$$

and the imaginary

$$C = \frac{cb - ad}{\omega(c^2 + d^2)}, \tag{12}$$

which are the constants sought. Before the parts of (7) can be separated it will be necessary to make some assumptions regarding Z. Perhaps the most general form which this impedance might assume in practice is that depicted below (Fig. 4). Here in series with the coil LR of resistance R and inductance L is inserted the resistance R'. Across the coil is connected a condenser equivalent to its distributed capacity. The capacity C' and conductance g' represent the possible connection of a second tube for purposes of amplification, the values thereof being suitably chosen by application of the input impedance equations to be developed. For the present for didactic reasons and in order to conserve space the form of Z will be restricted to a coil of inductance L and resistance R. If the coil is small and C is ignorable the series resistance R' may be included in R. If the coil is large, it will be necessary to discard R' in which case both C and C' may be identified with C_2. This then furnishes a case which is sufficiently general to lead to important and eminently practical equations. Special or degenerate situations in which any or all of the constants become zero will be considered later. The condition $Z = R + j\omega L$ substituted in (7) leads to:

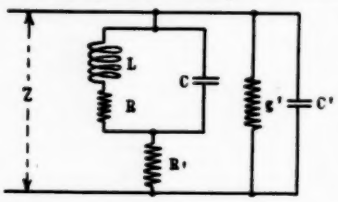

Fig. 4.

$$i_0 = \frac{\begin{array}{c} -\omega^2\{L[C_1 + C_m(1 + \mu_0)] + RR_0C_0\} \\ + j\omega\{R[C_1 + C_m(1 + \mu_0)] + R[C_1 + C_m] - \omega^2LC_0R_0\}e \end{array}}{\{R + R_0[1 - \omega^2L(C_2 + C_m)]\} + j\omega\{L + RR_0(C_2 + C_m)\}}, \tag{13}$$

from which:

$$\begin{cases} a = -\omega^2\{L[C_1 + C_m(1 + \mu_0)] + RR_0C_0\}, \\ b = \omega\{R[C_1 + C_m(1 + \mu_0)] + R[C_1 + C_m] - \omega^2LC_0R_0\}, \\ c = R + R_0[1 - \omega^2L(C_2 + C_m)], \\ d = \omega[L + RR_0(C_2 + C_m)], \end{cases} \tag{14}$$

where $C_0 = C_1C_m + C_2C_m + C_1C_2$,

which on substitution in (11) gives for the effective conductance:

$$g = \frac{\omega^2C_mR_0\{[\omega^2L^2 + R^2][C_m + \mu_0(C_2 + C_m)] + RR_0C_m - L\mu_0\}}{[R + R_0(1 - \omega^2L(C_2 + C_m))]^2 + \omega^2[L + RR_0(C_2 + C_m)]^2}. \tag{15}$$

Similarly by proper substitution in (12) we find the effective input capacity to be:

$$C = \frac{[(R+R_0)^2+\omega^2L^2][C_1+C_m(1+\mu_0)] - R_0(R+R_0)C_m\mu_0 + \omega^2R_0^2(C_2+C_m)\left[(\omega^2L^2+R^2)C_0 - L\left(\frac{C_0}{C_2+C_m}+C_1+C_m\right)\right]}{[R+R_0(1-\omega^2L(C_2+C_m))]^2+\omega^2[L+RR_0(C_2+C_m)]^2}, \quad (16)$$

where $C_0 = C_1C_m + C_2C_m + C_1C_2$.

It will perhaps be illuminating to consider the general form of the loci of these equations. For concreteness, it will be supposed that the tube used resembles electrically the type $VT-1$ (type "J") tube manufactured by the Western Electric Company, the characteristics of which

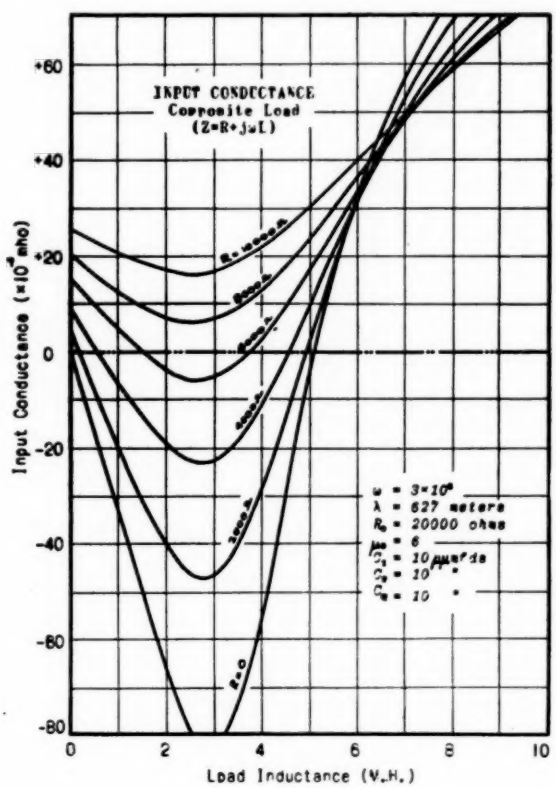

Fig. 5.

have been studied in an earlier paper.[1] From these experimental curves the value of μ_0 is about 6 and that of R_0 about 20,000 ohms at the most favorable points of operation. The tube capacities are not far from 10×10^{-12} farads. The extraneous capacities outside of the tube being

[1] Ballantine, Proceedings of the Institute of Radio Engineers, Vol. 7, No. 2, p. 129 (1919).

indeterminate will not be considered. Fig. 5 depicts the effective conductance as a function of L with R (the inserted resistance) as a parameter, for the above conditions and a wave-length of 627 meters represented by $\omega = 3 \times 10^6$. The effect of varying the inserted inductance and resistance and the fact that suitable values of L tend toward regeneration (a negative conductance) while other higher values and the insertion of resistance result in a loss (positive conductance) is well brought out. Several important practical deductions follow immediately from the relations exhibited by (15) and Fig. 5.

The occurrence of $\omega^2 C_m$ in the numerator suggests that in general the feed-back effects will increase with these factors. This is very probably the reason why this mode of securing regeneration is so eminently successful at short wave-lengths; also why many cascade high frequency amplifiers function as excellent power absorbers at these frequencies unless great care has been taken in their design. It is quite within the domain of possibility, in spite of a popular and most pernicious view, to design fairly efficient amplifiers for high frequencies if sufficient cognizance is taken of the relations (15) and (16). It is, however, beyond the scope of these remarks to consider the many and complex problems presented by amplifier design, the discussion of which will be reserved for a paper under preparation. That the increase in feed-back effects is rapid for increasing frequency is indicated by Fig. 6, in which a family of curves for several wave-lengths has been plotted.

The range of regeneration is determined by the position of the roots in the equation:

$$\omega^2 L^2[C_m + \mu_0(C_2 + C_m)] - \mu_0 L + R^2[C_m + \mu_0(C_2 + C_m)] + RR_0C_m = 0, \quad (17)$$

which is the numerator of (15) equated to zero. If the roots are given by:

$$L_{1,2} = \frac{\mu_0}{2\omega^2 C'} \pm \sqrt{\left(\frac{\mu_0}{2\omega}\right)^2 - RC'(R_0C_m + RC')}, \quad (18)$$

regeneration and a negative input conductance are available between L_1 and L_2 provided, of course, that they are real. This range in general becomes more restricted with increasing R until a unique value is reached above which the intersection is imaginary and there is no range. The necessary conditions for regeneration then, are defined by the discriminant of (17) as:

$$\mu_0^2 > 4\omega^2 C'R(C'R + C_mR_0), \quad (19)$$

where $C' = C_m + \mu_0(C_2 + C_m)$,

above which a negative conductance is impossible through any adjustment of L. An important practical deduction from this is that for a given frequency it is possible to load the plate circuit with an impedance for purposes of linking two tubes in cascade and by proper proportioning of R and L to have the input circuit exhibit no effect other than that of an increase in capacity. Under these conditions operation will be stable

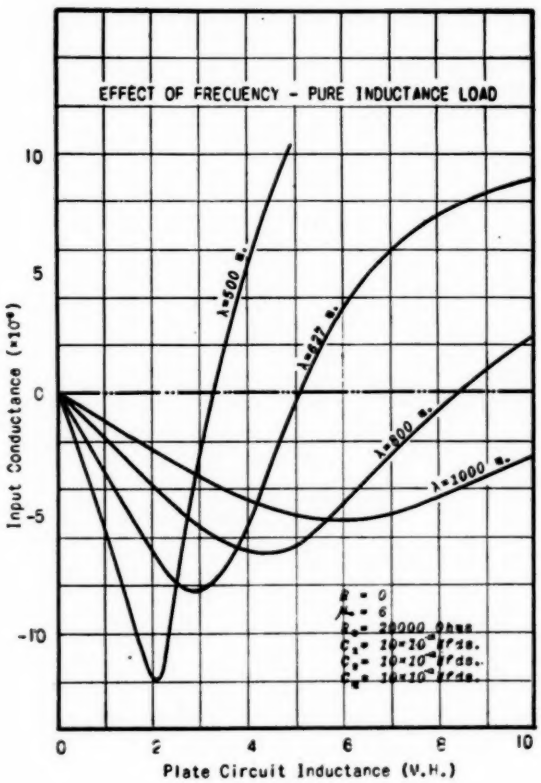

Fig. 6.

with freedom from howling and power-absorbing tendencies. Reserving further remarks upon this for a more pertinent occasion, let us consider a few significant special cases.

CASE I. PURE INDUCTANCE IN PLATE CIRCUIT.

The curves in Fig. 5 have indicated the justification for disregarding resistances of the order of 100 ohms at frequencies sufficiently high to render the use of a plate circuit inductance advantageous. The assumption then, that $R = 0$ covers the important practical case of an inductance coil with sufficient accuracy since the radio frequency resistance

of the usual inductor will be of this order. Any distributed capacity that the coil may possess may be conveniently incorporated in the discussion by adding it to C_2. This capacity will not, of course, be a constant but will vary with L if L is varied by means of the turns, but as shown by Hubbard[1] the change is not great after the first few turns. This is not a potent consideration. Introducing the proper conditions in (15) and (16) gives:

$$g = \frac{\omega^2 C_m R_0 \{\omega^2 L^2 [C_m + \mu_0(C_2 + C_m)] - \mu_0 L\}}{[R_0 - \omega^2 L R_0(C_2 + C_m)]^2 + \omega^2 L^2}, \tag{20}$$

$$C = \frac{[R_0^2 + \omega^2 L^2][C_1 + C_m(1 + \mu_0)] - R_0^2 C_m \mu_0 + \omega^2 L R_0^2 (C_2 + C_m)\left(\omega^2 L C_0 - \frac{C_0}{C_2 + C_m} - C_1 - C_m\right)}{[R_0 - \omega^2 L R_0(C_2 + C_m)]^2 + \omega^2 L^2}, \tag{21}$$

where $C_0 = C_1 C_2 + C_1 C_m + C_2 C_m$.

The roots of (17) are now:

$$L_1 = 0, \qquad L_2 = \frac{\mu_0}{\omega^2 [C_m + \mu_0(C_2 + C_m)]}, \tag{22}$$

and regeneration is possible in this interval. It may be of some interest and use to examine the conditions for maximum regeneration. For this purpose the *optimum* values of L are obtained in the customary manner by equating the derivative of (20) (L variable) to zero. The solution of the resulting equation defines this as:

$$L_{opt} = R_0 \frac{\omega R_0 C' \pm \sqrt{\omega^2 R_0^2 C_m^2 + \mu_0^2}}{\omega [R_0^2 \omega^2 (C_2 + C_m)(C' + C_m) - \mu]}, \tag{23}$$

which may be written in what may prove to be a more convenient form if we put $\rho = \mu_0 / R_0$ where ρ is the *mutual conductance*, a constant of the tube:

$$L_{opt} = R_0 \frac{\omega C' \pm \sqrt{\omega^2 C_m^2 + \rho^2}}{\omega \left[\omega^2 (C_2 + C_m)(C' + C_m) - \frac{\rho}{R_0}\right]}, \tag{24}$$

where $C' = C_m + \mu_0(C_2 + C_m)$.

The existence of a pair of roots suggests the occurrence of a maximum of positive conductance in addition to that in the regenerative region. Of these we will be chiefly concerned with that conjoined with regeneration obtained by taking the negative sense of the ambiguous sign in (24). Relations (22) and (24) will, I think, be found of some practical value in selecting proper inductance values for regenerative receiving circuits. Regarding receiving circuits it may be interesting to note,

[1] Hubbard, PHYSICAL REVIEW, Vol. IX., No. 6 (2nd Series), 1917.

parenthetically, that when L is very large the effective input conductance becomes:

$$g = \frac{\omega^2 C_m R_0[C_m + \mu_0(C_2 + C_m)]}{1 + [\omega R_0(C_2 + C_m)]^2}, \tag{25}$$

in which $\omega R_0(C_2 + C_m)$ can usually be ignored in comparison with unity giving very approximately:

$$g = \omega^2 C_m R_0[C_m + \mu_0(C_2 + C_m)], \tag{26}$$

indicating the loss due to a large inductance such as a pair of telephones of considerable distributed capacity. This result has a very significant bearing upon the theory of detection at radio frequencies and suggests as a fertile and practical field for research the necessary modification and expansion of current theories to include these phenomena. It would seem that some of the failures of the more modern tubes (even those containing suitable quantities of gas) might be attributed in part to their rather large inter-electrodic capacities. In any case it is quite apparent

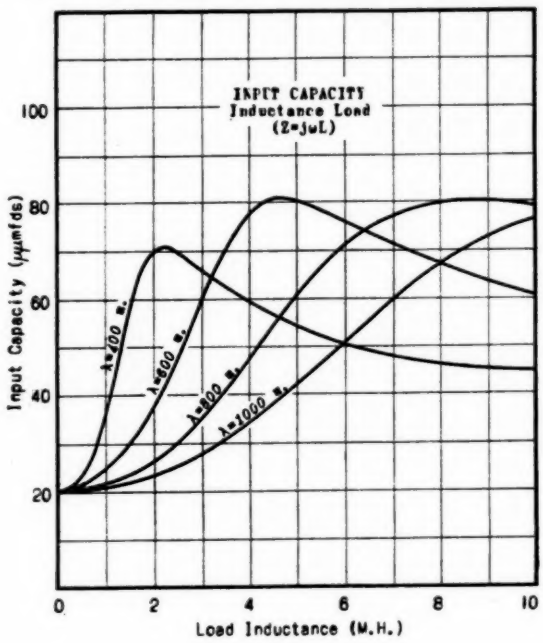

Fig. 7.

that equal consideration should be given in proportioning the electrodes to the attainment of low values of C_m as to obtaining favorable values of μ, ρ and R_0. An important reduction might be effected for example, by running the grid lead out through the wall of the tube rather than through the base and arranging the connecting wires to the external

circuit intelligently. There is no point, however, in tolerating a capacity of say 100×10^{-12} farads outside the tube and sacrificing something in μ, ρ and R_0 to effect a reduction of 5×10^{-12} farads inside. If feed back is desired, of course, the connecting wires should be arranged in the usual way.

The general form of the curves for the effective input capacity is shown in Fig. 7 for various wave-lengths and $R = 0$. Professor Morecroft's measurements were probably made at large wave-lengths so that the 1,000-meter curve is perhaps best suited for comparison (see capacity curve in Fig. 18, Professor Morecroft's paper).

CASE II. PURE RESISTANCE IN PLATE CIRCUIT.

As a further application of the expressions (15) and (16) to practical arrangements we may profitably consider the rather simple case of a pure resistance connected in the plate circuit. Pure, high-valued resistances, since the advent of the spluttered tungsten film are practically available. This constitutes an important practical case since such resistances are frequently employed as connecting links between the tubes in cascade amplifier systems. In this case a value of R may be used which represents the composite resistance of the load with the input conductance of the succeeding tube connected across it. The input capacity of such a tube may be conveniently included in C_2. Substituting $L = 0$, (15) and (16) yield without great difficulty:

$$g = \frac{\omega^2 C_m R R_0 [C_m(R + R_0) + R\mu_0(C_2 + C_m)]}{(R + R_0)^2 + \omega^2 R_0^2 R^2 (C_2 + C_m)^2}, \tag{27}$$

$$C = \frac{[R+R_0]^2[C_1+C_m(1+\mu_0)] - R_0(R+R_0)C_m\mu_0 + \omega^2 R^2 R_0^2 C_0(C_2+C_m)}{(R + R_0)^2 + \omega^2 R_0^2 R^2 (C_2 + C_m)^2}, \tag{28}$$

which may be placed in the simplified forms:

$$g = \frac{\omega^2 C_m Z [C_m + \mu(C_2 + C_m)]}{1 + \omega^2 Z^2 (C_2 + C_m)^2}, \tag{29}$$

$$C = \frac{C_1 + C_m(1 + \mu) + \omega^2 Z^2 (C_2 + C_m)(C_2 C_m + C_1 C_2 + C_1 C_m)}{1 + \omega^2 Z^2 (C_2 + C_m)^2}, \tag{30}$$

$$\text{where} \begin{cases} \mu = \mu_0 \dfrac{R}{R + R_0}, \\ Z = \dfrac{R R_0}{R + R_0}. \end{cases}$$

At sufficiently low frequencies terms involving ω may be disregarded

and these degenerate into:

$$\left.\begin{aligned} g &= 0, \\ C &= C_1 + C_m(1 + \mu). \end{aligned}\right\} \tag{31}$$

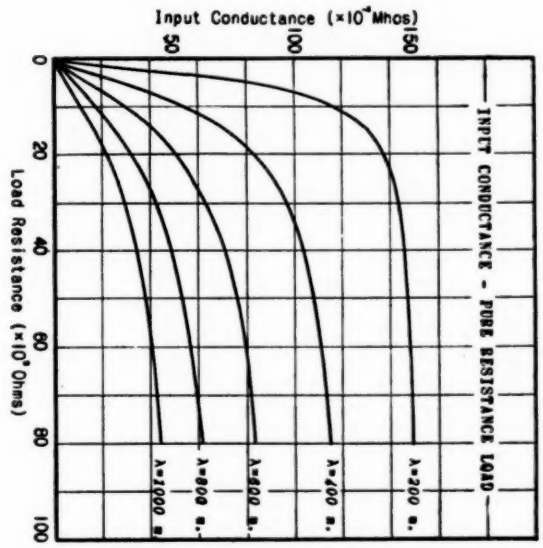

Fig. 8.

The general graphical form of these expressions is exhibited in Figs. 8 and 9 for the parameter ω and variable R. The variation of the effective conductance and capacity with R corroborates the physical expectations. In order to duplicate these curves experimentally some feasible means of maintaining constant the plate potential with varying values of R must be employed. This might consist, for example, in

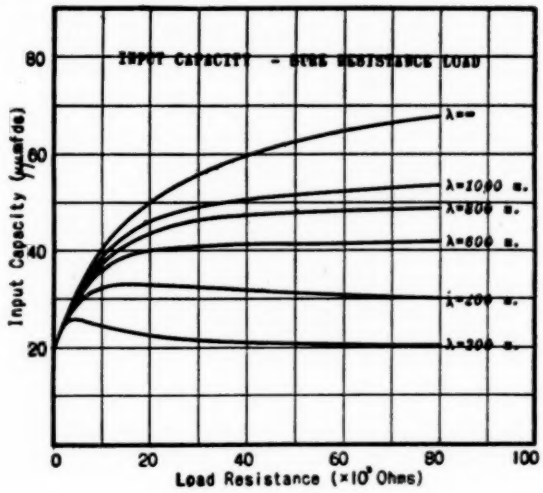

Fig. 9.

measuring the steady direct current through the tube, computing the drop through the resistance inserted and making suitable adjustments in the plate battery voltage. Otherwise the proper values of μ and R_0 must be substituted. The conductance in this case is always positive and represents a loss. The influence of the frequency is shown directly in Fig. 10 which represents the case of $R = 60 \times 10^3$ ohms ($\mu = 4.5$), a resistance suitable for voltage amplification. The reason for the failure

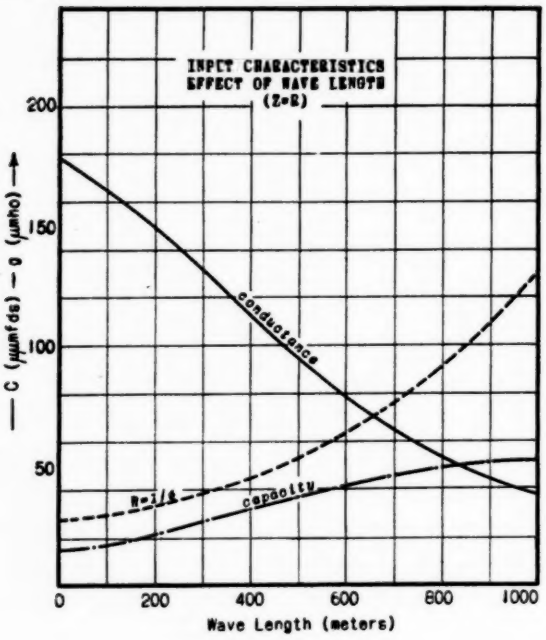

Fig. 10.

of the arrangement as an amplifier at wave-lengths of the order of 200 meters is at once evident for at this point the plate circuit resistance, across which the next tube is connected, is shunted effectively by a resistance of but 6,600 ohms due to the input conductance of this tube and a capacity of 31×10^{-12} farads due to C_2 and the input capacity of the tube. The drop across the resistance is then 3,000 i_p instead of 60,000 i_p and the arrangement is an amplifier in name only.

The relations developed in these paragraphs, particularly (15) and (16) will, I believe, be found sufficiently comprehensive to furnish the student and investigator of vacuum tube performance with an adequate foundation for further work on this subject and its manifold applications, among the most important of which are to be included the design of regenerative receiving circuits and of multi-stage amplifiers for radio frequencies. It is hoped, further, that certain of the expressions will be found of immediate practical value.

PHILADELPHIA, PA., November 12, 1919.

EXPERIMENTS IN BINAURAL PHASE DIFFERENCE EFFECT WITH PURE TONES.

BY MARGERY SIMPSON.

SYNOPSIS.

There are no published results which show the variation of angular displacement of the apparent source of sound with changes in phase difference at the ears.

Linear Relation between Angular Displacement and Phase Difference.—Tuning forks were used for the source of pure tones. Equal intensity was maintained at the two ears, while a difference in phase was secured by a difference in the lengths of the sound paths to the two ears. Results were obtained which are best represented by a linear relation between the phase difference and the angular displacement of the sound image or phantom from the median plane. The ratio of phase difference to apparent angular displacement varied with the frequency. A comparison of these ratios was made with the theoretical values as calculated by Stewart and Hartley, but the frequency range was not sufficient to enable final conclusions to be drawn.

THE purpose of this article is to give the results of certain quantitative measurements of the angle through which the sound image or phantom, as termed by psychologists, is displaced from the median plane for a given difference of phase in the sound at the two ears.

Method.—Tuning forks were used as the source of pure tones. The given difference of phase was secured and maintained by means of unequal lengths of sound paths to the two ears, while the intensities at the two ears were kept equal.

Apparatus.—The source fork was driven in tandem with an electrically maintained driving fork of the same frequency. The sound was conducted from the source fork through two brass tubes which were rigidly fixed to the iron mount of the source fork, and whose axes were perpendicular to the face of the vibrating fork. One of these, a telescoping tube, provided for the intensity adjustment, and carried a vernier sliding upon a millimeter scale. The driving fork and the source fork were each inclosed in wooden boxes, heavily lined with hair felt which served as sound screens. The sound was led from the brass tubes through equal lengths of rubber tubing to a second pair of brass tubes. One of these had a fixed length, and the other, a telescoping tube, provided for the phase adjustment and carried an indicator sliding upon a millimeter scale. A brass shut-off and reversing valve was attached to the ends of the tubes

last named. The sound was then conducted through equal lengths of rubber tubing and through stethoscope binaurals to the ears.

It is obvious that changing the length of one of the paths would, through changes in absorption and especially in resonance, produce a difference of intensity at the two ears. Hence, a calibration for equal intensity at the two ears was necessary. This was made for the frequencies 512 and 256 d.v. by means of Rayleigh discs. No Rayleigh disc was available for the 128 frequency. But inasmuch as experience with the 256 and 512 frequencies showed that the calibration was a refinement not warranted, on account of the inaccuracy of the observations and the very small error due to intensity differences, the more crude method of adjusting by the ear was employed. When the two intensities with equal phase produced no noticeable shift of the sound image from the median plane, the two were regarded as equal.

This method of determining the phase difference depends upon the velocity of sound. As the velocity is diminished in passing through tubes, it was calculated for the three frequencies, assuming the correctness of Helmholtz's[1] theoretical equation and using Seebeck's experimental value for the arbitrary constant. These considerations gave 339.4 meters per second for the velocity at room temperature. From this value conversion factors were obtained which express the degrees phase difference per centimeter of path.

Method of Observation.—A series of settings for varying phase differences, leading in phase at either ear, was given at random to an observer. The observer, who was seated at the center of a graduated arc 65 cm. in radius, localized the sound image in degrees right or left of the median plane. The operator opened and closed the shut-off valve at the speed desired by the observer, and continued this until the sound image was localized. The greatest difference in path was about one fourth wavelength, and the phase was made to lead in either ear by the reversing valve.

Results.—Experience with this apparatus showed that if a number of observations are made and plotted with the phase difference, ϕ, and the angular displacement, θ, as coördinates, the mean of the observations was best represented by a straight line passing through the origin. The experiment was conducted with the same four observers for each of the three frequencies 512, 256, and 128 d.v. Three of the observers described the image as distant; that is external to the head, and the fourth localized it as within the head and projected it upon the arc. This localization within the head seems unusual and undoubtedly lessens the reliability

[1] Barton, Text Book on Sound, p. 542.

of the results of this observer which show greater disagreement than those of the other three.

The image seemed most sharply defined for the 512 frequency. With the lower frequency the image was broad and especially so in the region of maximum displacement. The slopes of the curves mentioned were

TABLE I.

Observer.	$N=512$.		$N=256$.		$N=128$.	
	Number of Curves.	ϕ/θ.	Number of Curves.	ϕ/θ.	Number of Curves.	ϕ/θ.
S.....	9	2.69	7	1.55	8	1.27
H.....	5	2.15	2	1.09	1	.88
B.....	9	1.87	12	1.61	16	1.24
M.....	10	1.26	16	1.16	16	.97

determined and these values of ϕ/θ are reported in Table I. These experimental values of ϕ/θ for each observer are plotted as a function of the frequency. (Fig. 1.)

Discussion of Rresults.—The results obtained through the range of frequency 128 to 512 d.v. are in general agreement with those of previous

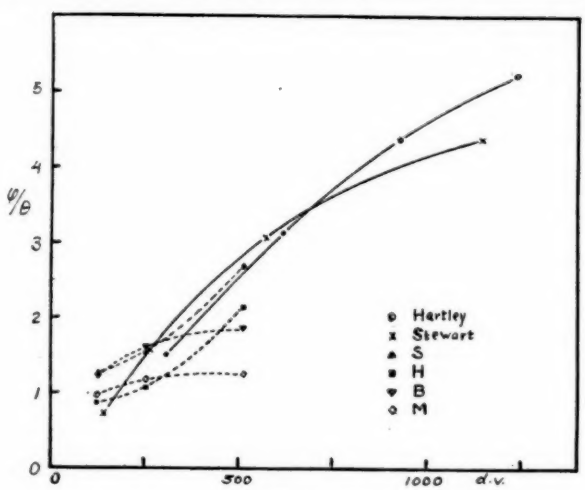

Fig. 1.

qualitative work as summed up by Stewart:[1] that phase difference in sound at the ears gives rise to a sense of localization; and that the phase difference necessary to produce a given displacement increases with the frequency.

Stewart[2] has calculated theoretically the phase difference at the ears

[1] G. W. Stewart, PHYS. REV., 9, 502, 1910.

[2] G. W. Stewart, PHYS. REV., 33, 467, 1911.

for varying positions of the source upon certain assumptions. He gives these calculated values for the frequencies 276, 553, and 1,106 d.v. and also plots a curve for the frequency 276 d.v.

Hartley[1] has extended these calculations to the frequencies 310, 620, 930, 12,40, and 1,860 d.v. His calculations are based upon assumptions slightly different from those of Stewart's.

These theoretical curves show that as the frequency increases, the phase difference necessary to produce a given displacement increases. The number of degrees phase difference necessary to produce a degree shift in localization for the various frequencies was determined from the theoretical curves by finding the slope through the origin. These values of ϕ/θ are plotted as a function of the frequency on the same graph as the experimental values. The close agreement of the theoretical and experimental results indicate that phase difference is an important factor in the localization of sound, at least within this range of frequencies.

The difference in the slopes obtained experimentally for the individual observers for each frequency indicates individual differences in the degree in which phase difference gives rise to localization. The value of ϕ/θ for each observer increases as the frequency increases, but not in the same ratio. The slopes for the individual observers varied from day to day and during the day, perhaps due to his physical condition.

These quantitative results would seem to support the direct perception of phase. But these results must be extended to other frequencies before any definite conclusion can be reached as to the exact rôle which phase difference plays in the localization of sound.

Summary.—This quantitative experimental work deals with the localization of the image of a pure tone with equal intensity and a phase difference at the ears. The results confirm in general those of previous investigators, within the range of frequencies 128 to 512 d.v., and may be stated as follows: phase difference gives rise to a sense of localization; phase difference is an important factor in localization; the value of ϕ/θ is a function of the frequency. The experimental values of ϕ/θ are in good agreement with the theoretical results of Stewart and Hartley.

I wish to express my acknowledgments to the members of the physics staff, and especially to Professor G. W. Stewart who suggested the problem and who acted as observer; to Mr. E. M. Berry and Mr. H. M. Halverson who acted as observers and to Mr. J. B. Dempster for his assistance in the construction work.

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[1] R. V. L. Hartley, PHYS. REV., 12, 373, 1919.

THE FUNCTION OF INTENSITY AND PHASE IN THE BINAURAL LOCATION OF PURE TONES. I.

BY G. W. STEWART.

PART I. INTENSITY.

SYNOPSIS.

Logarithmic Law.—The only published quantitative experiments giving the effect of intensity alone are those of Hovda and the author who found a "logarithmic law," viz., that the effect of intensity only was an apparent angular displacement from the median plane proportional to the logarithm of the ratio of intensities at the ears.

Logarithmic Law Extended; the Value of the "Constant."—The law just cited is found to be correct for the three frequencies 256, 512, and 1,024 d.v., and for a displacement range from 0° to almost 90°. But some individuals (two out of four) do not have a fused tone or phantom source at the frequency last named and indeed, at other frequencies also. The constant, in the logarithmic law, *i.e.*, the value of the displacement when the logarithm of the ratio of intensities is unity, is ascertained for several individuals at the above frequencies. For each individual the constant *decreased* with *increasing* frequency.

The experimental method is the one previously used. The logarithmic law is shown not to be an extension of Weber's law, well known to psychologists, but a new law.

Computed Values of Intensity Ratios for Given Positions of Source.—The actual intensities obtaining at the ears when a source of sound is placed at different positions relative to the ears are considered and the theoretical results are plotted and compared with the experimental results.

Intensity Not an Important Factor.—The wide divergence between experimental and theoretical values shows that an explanation in terms of intensity of the ability to locate a pure tone, 256 to 1024 d.v., is not possible. Intensity cannot be an important factor in localization of pure tones in this range of frequencies.

I. INTRODUCTION.

A SURVEY of the many reports of experiments in the localization of sound that discuss the function of intensity would not prove of sufficient interest here for, in every instance except those noted, the experimenters have not studied intensity effects with the assistance of quantitative measurements. Indeed, for the most part, writers have been content to discuss what appeared to them must be the effect of intensity differences at the ears. Rayleigh[1] was the first to point out that for frequencies of 128 and 256 d. v. the difference in intensity at the ears could not account for the ability to locate the source of sound.

[1] Rayleigh, Phil. Mag., 1907, 13, p. 217.

The first quantitative measurements of the intensity effect with a pure tone were made by the writer, and Hovda[1] who discovered that if relative intensities at the ears of a pure tone, 256 d.v., were varied, but with the phase difference maintained at zero, there resulted a displacement of the apparent source of the fused sound from the median plane, this displacement, θ, to the right, and the intensities at the ears, I_R and I_L, having the following relation,

$$\theta = K \log_e \frac{I_R}{I_L},$$

wherein K was a constant for an individual. This source of fused sound appeared to have a position external to the head in a plane approximately horizontal, usually in front rather than in the rear, and the displacement occurred in a circle of constant radius.

The purpose of the present paper is to present the effect of intensity only in producing a displacement of the apparent source, or the image,[2] from the median plane, to discuss these results in comparison with intensity relations existing when a similar source is displaced in a circular path about the head, to make note of the limitations of the logarithmic law just cited and to derive conclusions as to the importance of intensity as a factor in the location of pure tones of the range of frequency here considered.

II. APPARATUS.

The apparatus and method used throughout these as well as the previous experiments of Stewart and Hovda are similar in all essential respects and have received sufficient description in the article[3] already published. For the sake of clearness it should be briefly stated that the source of the sound was a tuning fork, that the observer used stethoscope binaurals and sat at the center of a circular scale in order to ascertain the angular displacement of the image, and that the intensities were varied by altering the position of one of the receiving tubes at the fork without producing a phase difference, and that the relative intensities at the ears I_R and I_L were ascertained by means of a Rayleigh disc.

III. RESULTS.

Values of K.—The procedure was to take at one sitting a number of observations of θ, the angular displacement, with known relative values of I_R and I_L, the intensities at the ears, and then subsequently to draw the straight line representing the mean of the observations and calculate

[1] Stewart and Hovda, Psych. Rev., XXV., No. 3, May, 1918, p. 242.

[2] The term "image" is here used as commonly in physics, *i.e.*, to refer to an apparent source of sound.

[3] Stewart and Hovda (loc. cit.).

the value of K. At least 24 to 50 observations were made for each curve, these being scattered so as to be "at random." In Table I. are shown the results thus far obtained, which, though not many, doubtless are sufficient to enable definite conclusions to be drawn.

TABLE I.

Frequency.	Observer.	Number of Curves.	Value of K.
256.....	S	3	16°
	B	5	30°
512.....	B	4	21°
	F	4	14°
	M	4	21°
1,024.....	B	8	10°
	F	8	7.8°
	M	8	18°

In Fig. 2 the straight lines represent the observations of B, the only experimenter who used all three frequencies. The observations of S are taken from the cited work of Stewart and Hovda, the 256 d.v. observations of B were taken by Mr. E. M. Berry in experiments conducted by himself and Mr. C. C. Bunch, the observations of B for the 512 and 1,024 d.v. were made by Mr. Berry in experiments conducted by Miss Caroline McGuire and the observations of F and M were also made with McGuire apparatus. In every case the linear logarithmic law seemed to hold Berry and Bunch showing that it held up to θ but slightly less than 90°. Two conclusions are to be drieved from the table, viz., that the constant K varies with individuals and that for an individual it decreases with increasing frequency.

Limits of the Application of the Logarithmic Law.—The first intimation that the law was not applicable to all ordinary frequencies with all individuals was found when the writer was unable at 1,024 d.v. to observe the rotation of the fused sound about the head with altered intensity ratios. This fact led to a brief series of experiments intended to ascertain at least some of the limitations of the applicability of the law. The apparatus, which is to be described in Part II. of this article, consisted essentially of a toothed wheel rotating in front of two bipolar telephone receivers. The currents produced actuated two head receivers and these, in turn, were attached in a suitable manner to stethoscope binaurals. With the apparatus adjusted for equal phase and intensity at the two ears, one of the rubber stethoscope tubes, right or left, was pinched, thus lessening the intensity on that side. Attention was confined to answering

the following questions, using a range of frequencies possible with the apparatus. Is there complete fusion and a rotation about the head of the image, apparently in accord with the logarithmic law? If partial fusion, where are the other images? Obviously, one could not test the law with such an apparatus quantitatively, but he could ascertain roughly at least some of the limitations of the applicability of the law. The purpose in these initial experiments was to find some of the limitations of the law and to point the way for future work. For the present purpose, then, it will be assumed that whenever a fused image moves from the median plane around to 90° right or left of that plane, the logarithmic law is followed. Clearly, when the image does move it is necessarily "fused" and hence in what follows the terms "fused image" or "complete fusion" or "incomplete fusion" imply an image moving as described. Only those frequencies were chosen which could be checked up by convenient tuning forks. Thus the frequencies 512, 640, 768, 896, 1,024, 1,280, 1,536, and 1,792 d.v. are the only ones referred to in this brief record.

With four observers *H*, *F*, *S* and *B* the following results were obtained. For *H*, with frequencies from 512 to 1,792 d.v. there was never complete fusion; from 640 d.v. to 1,536 there was no fusion; but below and above the two frequencies last named there was incomplete fusion. For *F*, up to 768 d.v. the fusion was complete, from 768 to and including 1,792 d.v. the fusion was incomplete. For *S* there is complete fusion up to 896 d.v., for 896 d.v. there was incomplete fusion, for 1,024 d.v. and 1,280 d.v. there was no fusion, and for 1,536 d.v. and 1,792 d.v. the fusion was practically complete. For *B* there were no frequencies possessing complete fusion, but there was a fused tone over the entire range and almost complete fusion at 1,792 d.v. These results are presented in Fig. 1, the two tones being represented by the single lines and fusion by the completeness of the connection between them.

When there was only partial fusion, one image did not rotate but remained directly in front, or, by trial, in the position determined by the difference of phase. It could, therefore, be ascribed to the phase difference effect. Thus the terms "partial" or "incomplete" fusion really mean fusion in more than one image. In the case where there was no fusion, and the tube at the left ear was being pinched the single image remained in front until the ratio of intensities was very large, *i.e.*, of the order of 200 with observer *S*, and then there appeared directly at the right, or approximately so, a second image which increased in clearness with further pinching, the image in front simultaneously disappearing, giving in the limit, as would be anticipated, but the one image to the right.

These experiments indicate that probably with most individuals (here four out of four) there is not complete fusion throughout all ordinary frequencies, but yet usually enough fusion to furnish a displaced image, that with some individuals (here two out of four) there are frequency gaps where the logarithmic law or any other law does not apply because of the absence of any fused image at all, and these gaps occur in the region having a frequency of the order of 1,000.

Another exception to the applicability of the law is the occasional serious deviation[1] from a straight line in attempting to represent a single individual's observations. Instead of a single straight line, required by the linear relation, the observations can, in these unusual cases, best be represented by two or three connecting straight lines of decidedly different slopes. The causes of such changes in the constant K at certain relative intensities are unknown.

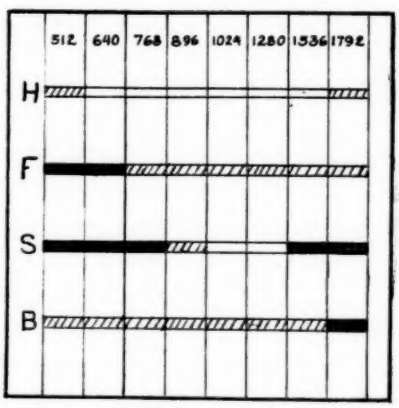

Fig. 1.

Intensity Combined with Phase Difference.—In actual localization there is a variation of both phase and intensity difference for a variation of position of the source, and it might be supposed that there might be found in the combination of the two factors an influence if the intensity factor were the only one. Yet such does not prove to be the fact. This was tested repeatedly, but not for all frequencies. Further reference will be made to these experiments in Part II.

IV. THEORETICAL RESULTS.

The logarithmic law as shown above, is in harmony with experimental facts, at least as high as 1,042 d.v. with some individuals. We should, in order to be able to determine the importance of intensity in the binaural location of a pure tone, determine the relation between intensity and angular displacement existing in actual experience. With the present stage of development in sound measuring devices, we must resort to theory for the results. Assume the head to be rigid sphere 60 cm. in circumference and the ears diametrically opposite one another. The computations for the intensities at the two ears with the source at certain different distances and having certain different frequencies have already

[1] Referred to in Stewart and Hovda article, loc. cit.

been published.[1] We select from available data the distance 477 cm. as a common one, though at distances of this magnitude or greater there is but little difference in relative intensities. The computations apply to wave-lengths of 120 cm., 60 cm. and 30 cm. and these are frequencies of 287, 574, and 1,148 d.v. at room temperature.

If the comparison is to be made with experiment, the values must be plotted in the same manner. Inasmuch as the experimental values will

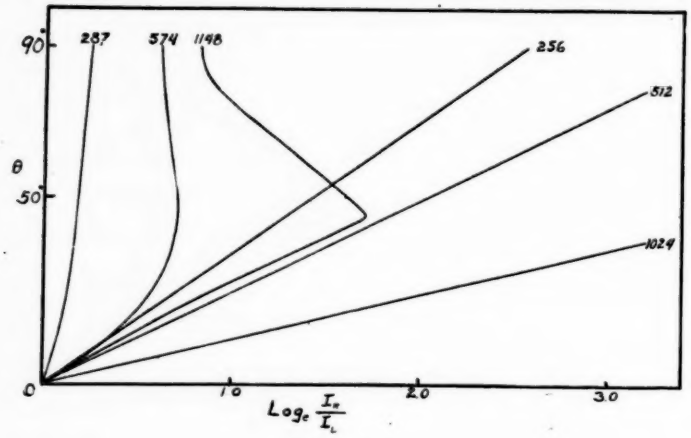

Fig. 2.

give straight lines only if θ and $\log_e I_R/I_L$ are used as coördinates, these are adopted in Fig. 2, the curved lines representing the computed values.

DISCUSSION.

Comparison of Theory and Experiment.—In Fig. 2, the experimental curves are not extended to angles of 90° though the logarithmic law holds approximately to that displacement, but are shortened in order to show the theoretical curves clearly without using too large a figure. Thus, in making a comparison, the reader should extend these straight line curves to $\theta = 90°$. Clearly there exists neither quantitative nor qualitative agreement between the two curves for similar frequencies. In fact, the theory shows the existence of two values of θ for one value of relative intensity, whereas experiment shows only single values. The disagreement, quantitative and qualitative, is so very great that intensity cannot be an important factor in the location of a pure tone. Moreover, inasmuch as the intensity when combined with phase difference does not seem to have an importance not possessed by intensity when the phase difference is zero, we must conclude that for frequencies of 256 to 1,024 intensity can be only a minor factor.

It may occur to the reader to question the soundness of the conclusion

[1] Stewart, PHYS. REV., XXXIII., No. 6, 1911, p. 467.

because of the fact that the two ears may differ in sensitivity to sounds. But a consideration of the nature of the experiments and of the logarithmic law shows at once that minor deafness in one ear would not change the slope of the curves, or the value of K, but would merely displace the curve along the logarithmic axis.

Errors.—It is not advantageous to discuss sources of error in the apparatus itself when, as shown in the cited article by Stewart and Hovda, the individual errors are relatively large. But these errors in no way weaken the conclusion given above, for the divergence between theoretical and experimental values are too enormous. An objection may be made to the use of the stethoscope binaurals which were inserted in the ears thus closing the external meatus, a condition which does not obtain in actual binaural location. This objection is removed by the experiments in the article last mentioned wherein it is shown that binaurals inserted give the same effects as open binaurals which do not close the external meatus.

Intensity Effect with Complex Tones.—The conclusions of our paper refer to pure tones only. When complex tones are used there enters a difference in the quality at the ears occasioned by the variation of the ratio of intensity at the ears with frequency. The effect of this difference in quality in assisting in localization is unknown and should be ascertained by experimental investigation. Moreover, there is also an effect upon quality determined by the position of the source, *i.e.*, around a corner, behind a building, in another room, etc. That this effect upon quality assists in determining location is a matter of common experience. Hence intensity does become, indirectly, an important factor in the location of complex tones.

Weber's Law.—In the former paper already cited, the logarithmic law was presented as an extension of Weber's law. But it is difficult for the writer now to accept this position. Weber's Law seems to be a law of the nervous system, and, in its integrated form, states that the response is proportional to the logarithm of the stimulus. Now the nerve response in each ear is to a first approximation, independent of what transpires in the other. The apparent displacement, θ, is not the effect of either nerve response alone, but, in fact, is the form taken by the recognition of the differences between the nature of these two nerve responses and hence is not at all a law concerning nerve response and stimulus as is Weber's Law.

The author wishes to acknowledge the invaluable assistance of Mr. A. E. T. Fant throughout these experiments.

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THE FUNCTION OF INTENSITY AND PHASE IN THE BINAURAL LOCATION OF PURE TONES. II.

BY G. W. STEWART.

PART II. PHASE.

SYNOPSIS.

Linear Relation; Phase Difference and Apparent Angular Displacement of the Source of Sound.—That a linear relation exists between phase difference at the ears and apparent angular displacement from the median plane between them, is demonstrated for pure tones within the range 100 to 1,200 d.v. Extended experiments are made with two individuals but additional evidence is adduced to show that the results have a general applicability.

Linear Relation; the Variation of the "Constant" with Frequency.—The "constant" in the above linear relation, or the slope of the curve with the two variables as coordinates, is approximately a linear function of the frequency.

Upper Limit of the Phase Difference Effect.—The upper limit of the phase difference effect seems to be from 1,000 to 1,500 d.v.

Double Images.—It is shown that the phase difference effect, if it be the controlling factor in localization, may produce multiple apparent sources. For a frequency of 1,024 d.v., the double phantom sources are found in actual experiment to have the relative positions indicated by the phase effect.

Intensity-effect and Phase-effect Combined.—It is found that when intensity difference and phase difference at the ears exist simultaneously, neither has a greater effect than when applied separately. Hence quantitative results obtained by using intensity difference only or phase difference only are transferable to the case of the combination.

Direct Perception of Phase.—Phase effect cannot be explained by a direct or by an indirect effect of intensity, for the phase effect at certain frequencies and with some individuals may exist when the intensity effect is wholly absent. Direct perception of phase is thus evident.

Theoretical Computation and Comparison with Experiment.—The phase difference at the ears for various angular displacements of the source from the median plane are computed for three frequencies scattered over the range under consideration. The relationship is not strictly linear but very roughly so. A comparison is made of the theoretical and experimental ratios of angular displacement and phase difference. There is a satisfactory quantitative agreement.

Phase Difference the Most Important Factor in Localization.—The significance of the quantitative agreement of experimental and theoretical results is presented and the conclusion derived that phase difference is the most important factor in localization of the source of a pure tone, 100 to 1,200 d.v. situated in a horizontal plane in front of the observer and 90° either side of the median plane. This conclusion is extended to include the region behind as well as in front of the observer. There are more factors in actual localization than in the case assumed in the theoretical discussion; for example, reflection and alterations in quality are doubtless important factors in the general case. Above the limit of the phase difference phenomenon, viz., approximately 1,200 d.v., intensity must become an important factor, but its importance varies with the individual.

I. INTRODUCTION.

THAT phase difference at the ears affects the localization of the sound source, has been known[1] for a number of years. Recently Hartley[2] has produced evidence in favor of the conclusion that the phase difference of a pure tone at the ears is the controlling factor in the location of such a sound source by an observer who holds his head stationary. The present paper will present a considerable addition to this evidence, making the conclusion just stated a fact to be accepted with full confidence.

In order that correct conclusions may be obtained in the problem of localization it is important that the simplest elements be involved in the first approximations. Thus the writer has used tones which were reasonably pure and has sought to ascertain not all the factors in localization, but rather the one clearly the most important.

II. APPARATUS.

The object of the experiments was to determine for various frequencies the relationship between the phase difference at the ears and the corresponding apparent angular displacement from the median plane. The observations were in part taken with a source of sound which for simplicity will be called a "phaser." The tones were not pure, but the results have been checked by additional experiments with tuning forks, as hereinafter described, and the use of the phaser may thus be considered as satisfactory for the purposes of these experiments.

The "Phaser."—This instrument consisted essentially of a rotating toothed wheel with two telephone bipolar receiver magnets placed radially close to the teeth and capable of being separated from each other by a variable known number of degrees of rotation of the wheel. Currents of the same frequency and complexity were incuded in the two bipolar receiver magnets, each of which was connected to a head receiver, and the phase difference of the currents could be controlled by the separation of the two bipolar receivers around the circumference of the rotating wheel. Thus the phase difference corresponding to any apparent angular displacement could be secured. A brief description of the phaser will suffice, for a study was not made to ascertain the shape of tooth giving the purest tone attainable or otherwise to produce the best instrument for the purpose. The motor was a small series one, designed for a rotary spark gap of a wireless set, and capable of operating continuously at 6,000 r.p.m. For securing fairly constant speed the field and armature

[1] For a review of literature see Stewart, PHYS. REV., Vol. IX., June, 1917, p. 502.

[2] Hartley, PHYS. REV., Vol. XIII., June, 1919, p. 373.

were operated on independent storage batteries. Toothed wheels of two, four, six, eight and twenty projecting teeth were used. The one with two teeth consisted, in fact, of a bar of iron approximately $9 \times 2 \times 0.6$ cm. The tone given by such a wheel is surprisingly pure and presumably so because the tones other than the fundamental are necessarily made up of intermittent vibrations and, as is well known, such intermittent tones are not readily recognized by the ear. For the same reason the tones produced by each of the other wheels were purer than one might anticipate.

By using in each circuit of the bipolar and head receiver an impedance having high self induction, the overtones were cut down relatively more than the fundamental. In the twenty-teeth wheel at high speed there occurred a tone having the frequency of the rotations per second of the wheel. This was practically eliminated by using a small capacity in series. But the excellent check on the phaser experiments with tones from tuning forks will be sufficient evidence that the results of the experiments are reliable and may be, so far as present accuracy is concerned, considered to be those obtained with pure tones.

Throughout the experiments with the phaser, in order to insure a sufficiently constant sound conductivity from each head receiver to the ear, stethoscope binaurals were inserted in the ears and were connected by rubber tubing to brass cylindrical plugs which fitted the openings in the head receivers.

The Circular Scale.—The observer sat at the center of a scale, 170 cm. in radius, prominently marked at 5° intervals up to 90° on each side of the zero which was directly in "front," or in the median plane between the ears. The scale was mounted at a convenient level, about that of the chin.

Tuning-fork-time-test.—Sounds from two electrically actuated tuning forks, slightly different in frequency, were led one to each ear by rubber tubing and stethoscope binaurals. As described in a former paper,[1] the apparent location of the sound rotates about the head with the changing of the phase difference of the two tones. By recording on a chronograph the time required for a rotation of the image[2] through an angle and comparing that with the time of a complete beat or phase difference variation of 360°, the phase difference corresponding to the apparent angular displacement from the median plane could be secured.

Tuning-fork-phase-difference-test.—In this test electrically driven tuning forks were used. The apparatus is that described by Miss M. Simpson.[3]

[1] Stewart, loc. cit.

[2] The word "image" is here used as in physics: it is the apparent source of the sound.

[3] Simpson, PHYS. REV.

With it settings of phase differences could be made at will and thus the relationship between these differences and the apparent angular displacements from the median plane could be ascertained.

III. Experimental Results.

Linear Relation.—That the apparent angular displacement from the median plane, θ, is a function of φ, the phase difference at the ears, was already well known, but the relationship between θ and φ was to be ascertained. This was accomplished as follows: The observer sat at the center of the scale and listened to the sound from the phaser. The assistant closed the circuits for two to five seconds and then opened them for one or two seconds' rest, repeating the operation until the observer signalled that he had determined the apparent location of the fused sound. Settings of the phaser were chosen so as to make the observations "at random." Fig. 1 shows the general nature of the results always obtained not only with the phaser but also with the tuning fork phase difference apparatus mentioned above. It is clear from Fig. 1 that a

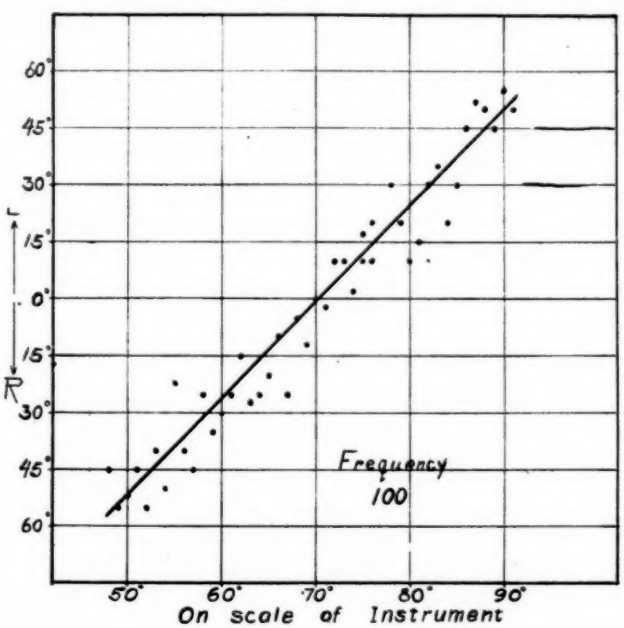

Fig. 1.

straight line will represent the mean of the observations more satisfactorily than any other curve. This was found to be the case at all frequencies. Hence the experiments were planned so as to determine merely the ratio φ/θ for each frequency, and this was accomplished by

plotting a curve from each observational "run" consisting of 16 to 20 observations. Since the completion of these observations another method of procedure, equally fair, has been found to produce much less individual error than shown in Fig. 1.

Values of φ/θ.—The values of φ/θ, or the ratios of phase difference to apparent angular displacement, are shown in Fig. 2.

The dots represent the values obtained from experimental curves

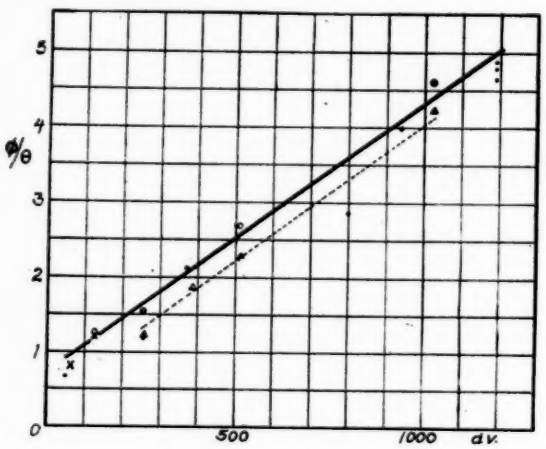

Fig. 2.

similar to that of Fig. 1, each involving 16 observations, the method being the use of the phaser with the additional aid of Helmholtz resonators presented to the head receivers and connected to rubber tubing leading to the stethoscope binaurals.

The circle with a dot center is the average of the values of φ/θ taken from four curves, the observations being made without resonators but with great care to secure a tone as nearly like that of a tuning fork as practicable by the use of inductance. Such a tone was not difficult to obtain for the tubber tubing caused rapid absorption of the overtones of this frequency of 1,024 d.v.

The crosses represent the observations taken by means of timing the rotation, the one for 64 d.v. being the average of 61 observations, and for 128 d.v. the average of 81 observations.

The small circles represent the average of curves similar to that of Fig. 1 obtained in the tuning fork-phase-difference-test. In fact, they are the observations reported by Miss Simpson.[1] The number of curves for the 128,256 and 512 d.v. were eight, seven and nine respectively.

The deltas represent observations taken by a second observer *F*,

[1] Loc. cit.

using the phaser, each point being the average value of the ratio taken from five curves. The method of observing was modified, the experimenter adjusting the phaser at will by a mechanical connection. The observations became more consistent without any other error being introduced.

It should be here recalled[1] that as the phase changes the sound travels about the head from the median plane in front and external to the head to the side leading in phase, then approaches and enters the ear, passing through the head quickly at 180°, then to the other ear and from thence around to the front along a path symmetrical to the one just described. This seems to be the movement of the image, but in the paper last cited the frequency used did not reach 500 d.v. The curve now shown, Fig. 2, shows that this complete movement is impossible for the higher frequencies. For, with the frequency of 1,024 d.v. the ratio of φ/θ is 4.4 which means that when φ is 180°, θ has become 41°, thus the image does not pass to the side of the head but, at $\varphi = 180°$, jumps from $+ 41°$ to $- 41°$, and then proceeds to 360° or 0° in a path symmetrical to the one from 0° to $+ 41°$. This jump has already been mentioned by Bowlker.[2] The fact that two different observers should obtain similar curves indicates the generality of the nature of the results. In all, four observers were tested for the extreme limits of the frequencies. The values of φ/θ for the four at 1,024 are 4.6, 4.5, 5.4 and 4.2. At the lower limit they were likewise in reasonably close agreement. Thus it is obvious similar curves would be had in general. One observer, *M*, a fifth, did not have a similar curve, the values obtained being reported by Miss Simpson.[3] But this observer was unusual in the effect of phase difference, being unable to get the image located external to the head.

There seems thus to be no reason to doubt the general conclusions from Fig. 2, though individual exceptions will be found. It should be pointed out that experiments thus far made indicate that the curve for φ/θ may bend sharply toward the origin at lower frequencies, but this region remains to be studied carefully.

The Frequency Limit.—By means of the phaser, observer *S* found that at 1,280 d.v. the phase difference continued to give an apparent displacement of the source, but that at 1,536 d.v. the source did not seem to move. All the observers tried, four in number, were found to have the same experience. Probably, then, this phenomenon ceases at from 1,000 to 1,500 d.v. with individuals generally. More[4] states that with

[1] Stewart, PHYS. REV., Vol. IX., June, 1917, p. 502.

[2] Bowlker, Phil. Mag. (6), XV., 1908, p. 318.

[3] Loc. cit.

[4] More, Phil. Mag., XVIII., 1909, p. 308.

1,024 d.v. his judgment became untrustworthy and that with 3,000 the effect ceased entirely. Rayleigh[1] gives his limit as 768 d.v. Myers and Wilson[2] state that with high frequencies the effect ceases. Preliminary experiments have shown the writer that the effect does not recur at frequencies of 2,000, 3,000, and 3,500 d.v.

Double Images.—It will be noted that, as already described, for 1,024 d.v., when φ has reached 180°, θ has become 41°. Experimentally, at 180° one should then have two images, + 41° and − 41°; and this was found to be the case. But if φ is made greater than 180°, will there continue to be two images, one being between $\theta = 0°$ and $\theta = -41°$, and the other between $\theta = +41°$ and $\theta = +90°$? Such was found to be the fact, but the distinctness of the second image differs with the observer.

In view of the above result a test was made in the open as follows: A blind-folded observer seated himself upon the ground and located without rotating his head, an electrically driven tuning fork, mounted on a resonator and situated on the ground at selected points of a semicircle of five meters radius. Experiments were conducted in such a manner that the only assisting sound was the pure tone of the tuning fork. For example, during transport from one point to another, the electromagnet operating the fork was short-circuited and the observer covered his ears. The fork had 1,024 d.v. and was driven in tandem by an electrically driven 512 d.v. tuning fork. The ground was heavily grassed, the observer held his body and head strictly stationary; the source was placed close to the ground and every precaution taken to avoid the influence of reflection. The typical results obtained are shown in Table I. A similar experiment by observer *H*, gave an average of 96° and, by observer *F*, a 90° separation. Observer *F* found a similar average with the phaser of 88°. In the 16 observations in Table I., there were four cases where the observer was incorrect in his impression as to which of the two locations was actually that of the source.

The significance of these outdoor experiments will appear in a later portion of this paper.

Combined Effect of Phase and Intensity.—In Part I. of this report was discussed the experimental effect of intensity only and in Part II. the chief variable is phase only. If either intensity or phase has any additional effect when in combination with the other, this should be ascertained. The following method was adopted. The phaser was set for a frequency of 512 d.v. At this frequency, according to the theory referred to in Part I., the ratio of intensities at the ears would be 0.6

[1] Rayleigh, Phil. Mag., XIII., 1907, p. 214.

[2] Myers & Wilson, Proc. Roy. Soc., LXXX., 1908, p. 260.

TABLE I.

Position of Fork.	Position of Image at Right of Front.	Position of Image at Left of Front.	Separation of Images.
40° L	20°	50°	70°
80° L	10°	80°	90°
50° L	20°	60°	80°
20° L	—	15°	—
30° R	20°	70°	90°
60° R	60°	50°	110°
90° R	90°	10°	100°
10° R	0°	0°	—
70° R	65°	30°	95°
0°	10°	—	—
30° L	45°	40°	95°
90° L	15°	80°	95°
50° L	15°	70°	85°
10° L	—	5°	—
40° R	40°	40°	80°
60° R	50°	50°	100°
			Av. 91°

for a displacement of 90° from the median plane. Using the experimental value of $K = 14°$, for observer F at approximately this frequency, it is found that a ratio of 0.6 would actually produce a displacement of 7.5° from the median plane. One of the tubes leading to the ears was pinched so that a displacement of approximately this amount was produced. Then a series of observations with varying phase difference was made with this selected intensity ratio. If intensity had any peculiar effect when in combination with phase, then it would be expected that a curve between phase difference and angular displacement would not be the same as that with equal intensity. But experiment showed that the same curve was always produced, though of course shifted on the θ axis by the same amount at every point. In other words, a difference in intensity merely caused a constant shift in the displacement θ. This experiment was repeatedly tried with differences of intensity causing a shift as great as 15° and yet without altering the nature of the phase difference curve. The ratio φ/θ was unchanged in every case. Thus neither intensity nor phase has any peculiar effect when in combination with the other and our results for either acting alone are to be considered as correct in all cases with pure tones. Thus although one may produce at the ears the same intensity ratio and phase difference as that occurring in actual localization, the effect of intensity is practically nil and the location is determined chiefly by phase difference.

IV. THEORETICAL CONSIDERATIONS.

φ a function of θ.—The results of computations from theory of the phase difference at the ears with the source in a horizontal plane and at an angle of θ from the median plane, were first presented by the writer.[1] For clearness the original curve is herewith reproduced as Fig. 3.

The ordinates, θ, refer to the angular displacement of the source of

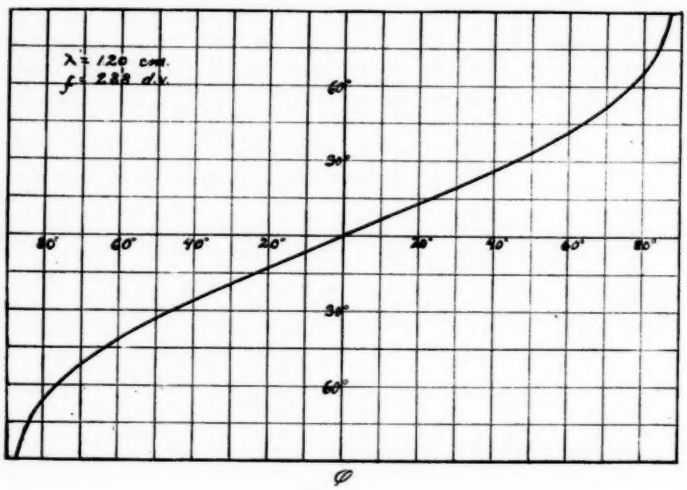

Fig. 3.

sound from the median plane, and the abscissæ, φ, refer to the phase difference at the ears. The computations assume the source at a distance of 477 cm., the wave-length 120 cm., the head a rigid sphere 60 cm. in circumference, and the ears diametrically opposite. The similarity of the curve to that obtained experimentally as in Fig. 1, is obvious.

Values of φ/θ.—The question arises as to whether or not the values of φ/θ for the theoretical case of the source actually displaced at an angle θ, correspond to the experimental values where φ is produced and θ chosen. If the values are in agreement, it would seem that the cases are similar, and this means that in the location of a sound source without head rotation, the phase difference is the important factor.

In order to compare the theoretical and experimental values, certain approximations were made. The experimental curve is linear so far as observations could determine. The straight line drawn was influenced largely by the observations of θ having a mean of 25°. Hence the theoretical values of φ/θ to be used in comparison should be those for θ about 25°. These values do not differ from $d\varphi/d\theta$ by more than nine per cent. on the average.

[1] Stewart, PHYS. REV., N. S., Vol. IV., September, 1914, Fig. 3.

The theoretical values were obtained from the results published in the article last cited and from the curves of Hartley[1] who assumes the circumference of the head 55 cm. and the ears 165° apart and the source of sound at a great distance. They are plotted as indicated in Fig. 4. The curves drawn are those of Fig. 2 and represent the experimental results. It is obvious that the theoretical and experimental values correspond remarkably well, the nature of the results being considered.

Intensity Effect.—In Part I. of this report the results of a study of the intensity effect, with frequencies 256 to 1,024 d.v., have been presented and the conclusion definitely drawn that intensity differences at the ears cannot be important factors in localization over this range of frequency.

V. Discussion and Conclusion.

Significance of Fig. 4.—Two obvious factors in localization are found in the differences in phase and intensity at the ears. The remarkable quantitative agreement between theory and experiment shown in Fig. 4

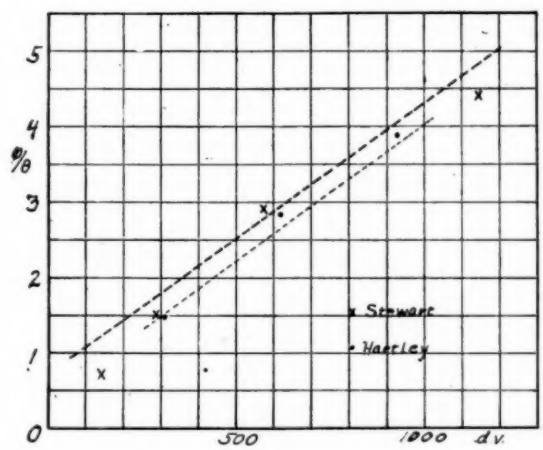

Fig. 4.

has the following significance: the angular displacement from the median plane produced by a given phase difference at the ears proves to be the very position for the source of sound that will, theoretically, produce the aforesaid phase difference at the ears. This quantitative agreement, although approximate, is highly satisfactory if appropriate allowance is made for the nature of the experiments. The conclusion must be that the phase difference is a factor of high importance in the frequency range considered.

The experiments of Bowlker[2] are in close agreement with Fig. 2. He

[1] Hartley, Phys. Rev., Vol. XIII., June, 1919, p. 373.

[2] Bowlker, loc. cit.

found that a wave-length of 36 inches was the one giving a displaced image of 90° for a difference of 180°. This would be a frequency of about 380 d.v. Reference to our curve, Fig. 2, shows that the frequency in our experiments would be 352 d.v. for one observer and 440 d.v. for the other. Bowlker used organ pipes for the source and hearing tubes $2\frac{1}{4}$ inches in diameter. With such a narrow aperture the relative intensities in the two tubes, one to each ear, would be fairly independent of the directions of the axes, and thus he was dealing with practically equal intensities as in our experiments.

Inasmuch as in Part I. of this report, intensity was found not to be an important factor in the binaural location of pure tones, our conclusion as the high importance of phase, is given additional weight.

Extension of Limited Application of Experimental Curve.—The results as presented in Fig. 2, do not cover all values of displacement either side of the median plane, but strictly speaking only up to the values of displacement for which the phase difference is 180°. Does phase difference remain the most important factor for these higher frequencies, where, as we have shown there is more than one image? Since the multiple images are found both with the phaser and in the open in actual localization, the additional factor of intensity entering the latter case does not modify or alter the effect of phase and hence its importance. In other words, phase difference remains the most important factor in localization throughout the range of frequency considered and for displacements of 90° from the median plane. Moreover, computations from the theoretical values of Hartley and of the author already referred to show that the separation of the images for 1,024 d.v. is approximately that obtained in the open, viz., 90°.

The evidence in hand both experimental and theoretical seems to establish beyond reasonable doubt that localization in the horizontal[1] plane for pure tones of 100 to 1,200 d.v. depends almost entirely upon phase difference at the ears. But the above discussion refers to localization within the region 90° on either side of the median plane and in front only. This first approximation in the explanation of localization need not be considered as limited to the front region only, for the well known confusion between locations in the front and in the rear indicates that the discrimination between such positions symmetrical to the median plane, rests upon additional factors probably of secondary magnitude. Indeed, many observers find that the difference of phase at the ears causes them

[1] All of our experiments refer to the horizontal plane, but a note should be made of the fact that in the experiments with the phaser and the forks the rotation was usually not strictly in a horizontal plane but could be made so by a slight forward tilt of the head.

to locate the source in the rear rather than in front. For the present, then, we are to assume that the most important factor in the frontal region is also the most important factor in the localization in the rear, leaving for future consideration the factors that distinguish between the positions symmetrical with the line connecting the ears. It is to be admitted freely, then, that the present paper does not offer a complete explanation of localization, but rather proves that the difference of phase phenomenon is the most important single factor in sound localization of a pure tone in a horizontal plane.

Complexity of Factors in Actual Localization.—One might decide that, since the only physical factors in a pure tone of given frequency are phase and intensity, and since we have all but eliminated intensity as a factor in localization, the only important factor left is that of phase difference. But this cannot be true in the sense that the phase difference is that produced merely by a single source and diffraction about the head as a sphere. For there are always present reflecting surfaces which are extensive enough to produce images. This is especially true of frequencies of the order of 1,000, having a wave-length of 34 cm. Although the effect at the opening of the external meatus is still expressible in phase and intensity, yet, in contrast to the case of a simple source, we have, in general, the equivalent of several sources, with most of them on the same side of the median plane as the source. Now while it has not been demonstrated that the two ears have a "resolving power," yet there would result from reflections an apparently diffused source of sound instead of the original source only. Consequently the observer could distinguish between a location on the right and left side of the median plane. Thus, assuming that phase difference is the most important factor in localization, it by no means follows that the case is as simple or needs to be as simple as that of a source and a rigid sphere with the two ears located diametrically thereon. The complexity of conditions involving reflection gives the single factor, phase difference, a greater opportunity to secure accurate location than die the simple theoretical case exist.

That reflecting surfaces are important was readily shown by the effect of rotating the head when attempting to locate the source in the open air experiments described. A slight rotation would sometimes shift the apparent location by as much as 20°. Doubtless there are other factors which enter into the localization of a pure tone. There are at least two additional factors in the case of a complex tone; they are the difference in quality at the two ears produced by diffraction, and the difference in quality of the sound arriving, depending upon the location of the source.

Intensity a Factor.—If the frequency exceeds the limit of phase difference sensitivity, our chief factor, phase difference, ceases to exist and we are dependent upon intensity effects, including not only the intensities produced at the ears by the source, but also by the mirror images. As elsewhere recorded, the writer has found that the effect of intensity-ratio as embodied in the formula presented earlier in Part I. of this article, becomes seriously modified in a limited region of frequency. With some individuals, for this frequency band, there is no displacement law as above given. For others, there ceases to be complete fusion in this region, though it is possible to locate a fused tone as well as to recognize another image at the same time. But it is not the purpose of the present article to cover the region above 1,200 d.v. as this will be left for further experimentation.

The Ear as a Physical Instrument.—Physicists are interested in the ear as a physical instrument and one might well inquire as to whether or not the experimental curve here shown can indicate any mechanical requirement of the ear. If the curve had been a straight line passing through the origin, it might then be said that, since equal time intervals correspond to phase differences that are proportional to frequencies, and since φ/θ would then be proportional to frequency, therefore equal values of θ would be obtained by equal time intervals. But the curve, if linear, does not pass through the origin and equal time intervals correspond approximately to equal angular displacements only at the higher frequencies. Hence we cannot say the angular displacement is caused simply by a time interval. It is more reasonable to assume that the phenomenon we are discussing is not one of the ear regarded merely as a mechanical instrument, but of the entire phenomenon of hearing.

Phase Difference Phenomenon Independent of Bone Conduction.—In binaural beats with a frequency of from 40 to 400 there is a complete rotation around and through the head of the fused sound as φ changes from 0° to 360°. This has been fully discussed and explanations offered,[1] but one writer[2] has interpreted this explanation as requiring cross-conduction for the phenomenon of localization of sound. This is incorrect and an additional word is necessary. As the fused sound rotates from $\theta = -90°$, through $\theta = 0$, to $\theta = 90°$, the location of the image is external to the head just as in the usual localization of a source of sound. But during much of the other half of the cycle the image is within the head and the experience is wholly different than in ordinary localization. The writer did use cross-conduction to explain this internal half of the

[1] Stewart, PHYS. REV., Vol. IX., 1917, p. 514.

[2] Hartley, loc. cit.

cycle but in no wise does this mean that cross-conduction enters into the act of localization of sound. In fact, as already indicated, we are interested only in the portion of the rotation $\theta = -90°$ to $\theta = +90°$, anticipating that the chief factor in localization in this region will prove to be the chief factor in general.

Perception of Phase.—As has been mentioned in various articles referring to perception of phase, many psychologists assert that the response of a sensory nerve is independent of the nature of the stimulus and if this is true, phase relations could not be found in the nervous impulses. Thus direct perception of phase could not exist. It would seem that the simplest interpretation of our experiments is the direct perception of phase. An effort has been made by some to explain the apparent perception of phase by the indirect effect of intensity arising from bone conduction between the ears. It would seem that the intensity effect is too small to warrant such an explanation. But what appears to be a crucial experimental test has been found. As recorded in Part I., observers *H* and *S* could not get any intensity effect whatever at, 1024 d.v., yet the same observers responded to phase difference at this frequency without any hesitancy. Thus, the effect of phase difference cannot be explained by any indirect intensity effect. It would thus seem necessary to admit direct perception of phase, at least until a possible different interpretation of the effect of phase is secured.

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THE INFLUENCE OF BLOWING PRESSURE ON PITCH OF ORGAN PIPES.

BY ARTHUR C. LUNN.

SYNOPSIS.

This paper suggests that the rise of pitch with increase of blowing-pressure may be due to the kinetic stiffness of the blowing jet or "air-reed," and the consequent discontinuity of pressure at the mouth due to the centrifugal force of curvilinear flow. Formulas are developed for the corresponding end-condition and for the relative increase in frequency of vibration, in terms of effective pressure and proportions of pipe and mouth, which agree approximately with the observed effect in a rough trial made. The simplifications introduced are discussed and certain contrasts suggested with the phenomena in tanks of water studied by Mallock. Experimental material for systematic comparison with the theory seems to be not yet available.

THE rise in pitch of an organ pipe with increase of blowing pressure is a familiar phenomenon, but seems as yet to have received no satisfactory quantitative explanation. The complete theory of the initiation and maintenance of the tone is recognized to offer a complex problem in the hydrodynamics of a compressible and viscous fluid, but the oscillations of the blowing jet are surely a main feature and in fact have given rise to the notion of the "air-reed," which has come to be a common mode of viewing the matter. Arguments for and against this notion are many, and suggest that the jet may act in some respects like a metallic reed for instance but in other ways fail to complete the resemblance.

The motion of a metallic reed, even when the tone of the complete pipe obtained by combination with a resonating tube is rich in harmonics, is known to be usually approximately simple harmonic. This suggests that the tongue of the reed can for a first analysis be viewed as a system with one degree of freedom, with inertia, friction, and stiffness, coupled with another system, the column of air, of more complex character; sometimes the "boot" of the reed also must be considered to partake of the coupling.

The recent vital improvements in tone of reed-pipes obtained by loading the tongues illustrate the importance of their inertia, and in most cases the stiffness of the tongue is a controlling factor. That the tongue itself offers enough internal viscosity to be of much consequence is doubtful, but the viscosity of the air flowing about it may lead to a

simulated damping-factor of the tongue itself which is probably of some influence.

In the case of the flue pipe the thin jet of air can hardly be conceived to possess sufficient inertia in line with that of the pipe-column to be of much relative importance, at least as to its variations with blowing pressure. The viscosity of the air is certainly a fundamental factor, perhaps more important in the initial "bite" of the wind-sheet than in the maintenance of the tone, but its effects are so distributed that they can hardly be allowed for by a simple damping-factor for the oscillating jet as a whole. On the other hand, the kinetic stiffness of the jet, due to its momentum transverse to the line of vibration, may very well be in its main features like the stiffness of a metallic reed.

Accordingly, the suggestion to be developed here is that for a first approximate treatment of the problem the end-condition for the mouth of the pipe should take account of such a discontinuity of pressure in passing across the jet as is needed to balance its kinetic stiffness, or the centrifugal force of its flow, as its curvature alters. For convenience of picturing the jet may be thought of as blowing across the end of the pipe-column, though the practical construction usually puts it at the side. To complete the simplification to a one-dimensional problem it will be assumed that the stream-lines of vibrational flow in the region near the jet are invariable, so that the "conductance" of this region of departure from plane waves is independent of the amplitude of vibration. Also it will be supposed that the wind-sheet at each instant can be regarded as of uniform thickness a and curvature σ.

In view of the other approximations the jet arc may be regarded as agreeing substantially in form with the parabola of equation $\xi = \frac{1}{2}\sigma\eta^2$, so that the volume of air momentarily displaced from within the pipe through a rectangular mouth of width w and height h is

$$w\int_0^h \tfrac{1}{2}\sigma\eta^2 d\eta = \tfrac{1}{6}w\sigma h^3,$$

giving as one condition

$$\tfrac{1}{6}w\sigma h^3 = -Sy \text{ at } x = 0, \tag{1}$$

in which S is the cross-section area of the tube, while x and y are equilibrium abscissa and momentary displacement of a cross-section of the contained gas. Here σ is taken as positive when the jet bends out, and x, y positive within the pipe in a direction away from the mouth.

The other relation required is that describing the balancing of centrifugal force in the jet by an excess of pressure on its inner side, positive or negative, and is therefore

$$a\rho v^2\sigma = -\kappa\frac{dy}{dx} \text{ at } x = 0, \tag{2}$$

in which p denotes density and κ volume elasticity of the gas, so that $\kappa = \gamma p$; where p is normal pressure and γ the ratio of specific heats.

Since $\gamma p/\rho = c^2$, where c is the velocity of sound, the elimination of σ from these two equations gives the required end-condition in the form

$$\frac{dy}{dx} = \frac{y}{b} \text{ at } x = 0, \qquad \text{where } b = \frac{c^2 w h^3}{6v^2 Sa}. \tag{3}$$

This b, of the nature of a length, is the characteristic coefficient in the theory, and in practical cases is large compared with the wave-length. The limiting case where b approaches infinity gives the familiar end-condition of elementary theory, according to which an anti-node of displacement occurs at the mouth. Inspection shows that the introduction of the modified condition given by (3) indicates a shifting of the anti-node into the pipe by an amount which increases as b diminishes; in other words the wave-length decreases as the jet-velocity v increases, which is qualitatively the effect to be explained.

For the case of a pipe maintaining a simple-harmonic vibration of wave-length λ the general condition indicates for the displacement s of the anti-node the equation

$$\tan \frac{2\pi s}{\lambda} = \frac{\lambda}{2\pi b},$$

which, when s is supposed small, reduced to the practical formula

$$s = \frac{\lambda^2}{4\pi^2 b}. \tag{4}$$

If end-conditions at the opposite end of the pipe, of length l, are constant the relative rise in frequency of vibration is therefore

$$\frac{\Delta n}{n} = \frac{\lambda^2}{4\pi^2 bl}. \tag{5}$$

This form is adapted for application to either open or closed pipes, speaking the fundamental or overblown, provided the proper value of b is taken in each case. For example the result may serve as a guide toward explaining the known differences of sensitiveness of pipes of different proportions to variations of bellows-pressure. It is not suggested however that any such simple formula is likely to apply well to the curious phenomena that occur at pressures below the minimum at which the wind begins to "reach" so as to produce a fully developed and stable tone.

Rayleigh[1] gives a table of observed increase of frequency with rise of pressure but does not fully describe the pipe used and indicates no theory.

[1] Rayleigh, Phil. Mag., 13, 340–342, 1882; Scientific Papers, II, 95–97, also 219, 220.

Finding no published data sufficiently detailed to permit comparison with formula (3) the writer made a rough test with a small diapason pipe, where the pitch rose about a semitone for increase of pressure from five to ten centimeters of water-column. The computed value, using for a the actual width of the wind-way and for v the value corresponding to adiabatic efflux under the observed bellows-pressure, was about 50 per cent. higher. This computation however overestimates the value of a, because of the vena contracta effect, and also of v, since because of frictional losses the pressure in the foot of the pipe is less than the bellows-pressure and is subject to further discount at the windway. It seems possible also that larger relative losses of pressure by friction and eddies at higher pressures may explain the fact that successive equal increments of bellows-pressure give decreasing increments in frequency of vibration.

It is of course not to be expected that the factor here regarded as primary should prove to be the only one included in an elaborate treatment by hydrodynamical theory. Some other factors that may be of influence have been mentioned for example by Mallock,[1] in connection with his experiments on surging oscillations supported by jets in a water-tank. It is not clear, however, that the analogy of this phenomenon with the sounding of an organ pipe is as close as Mallock seems to suppose. His apparatus is symmetric with respect to the jet, and the oscillation is mainly a variation of level under gravity, with compressibility probably of minor importance, in both respects contrasting with the case of the organ pipe; so that relations of phase at least might be quite different in the two cases.

In practice the problem is further complicated by the effect of the "nicking" on the values of a and v, and possibly by the influence of the "ears" and other features. In spite of the obviously crude nature of the analysis here given, however, the result seems to be near enough to indicate that the formula merits experimental test, including the matter of phase relations in jet and body of pipe.

THE UNIVERSITY OF CHICAGO,
January 10, 1920.

[1] Mallock, Proc. Roy. Soc., 95, 99–106, 1918.

THE SKIN EFFECT AND FLUX DISTRIBUTION OF CONDUCTORS IN PROXIMITY TO IRON MASSES.

BY A. PRESS.

SYNOPSIS.

Insufficiency of Arnoldian Theory of Slot Self-induction.—The hitherto accepted method for estimating self-induction of conductors in proximity to iron, which assumes that internal self-induction flux is short-circuited by the neighboring iron, so that in slot wound conductors all the internal self-induction flux passes across the slot, is disproved by test with a small exploring compass needle. It is more nearly true that the internal self-induction flux distribution of a conductor is much the same as in air and that practically it is only the external self-induction flux that takes advantage of the proximity of iron masses.

Comparison of A. B. Field's and the Author's Solution on the Basis of the Arnoldian Theory.—A correction needs to be introduced into A. B. Field's mathematical solution for the slot wound conductor problem. The two boundary conditions should be: first, that the value of H at the top of the mth conductor is due to the m conductors carrying the load current; and, secondly, that the value of H at the bottom of the mth conductor on the basis of the Arnoldian theory should be due to $(m - 1)$ current carrying conductors. When these boundary conditions are properly introduced, it is found that the skin-effect factors are very different to those derived according to the A. B. Field requirement in which one of the boundary conditions above referred to is replaced by the condition merely that the mth conductor shall carry the total line current.

Consequences of the Newer Theory.—For purposes of estimating the skin effect of conductors in iron slots, or near iron masses, results more consistent with facts will be obtained if the author's solution for the skin effect is massed rectangular conductors in air is assumed. However, in estimating the external self-induction the Arnoldian theory will still apply.

IN the number of the PHYSICAL REVIEW for October, 1916, the writer gave the solution of the "Skin Effect" problem of massed rectangular conductors in air. At that time it was not thought that the above solution really applied equally well to the problem of the skin effect of conductors occurring in the open type of slots usual in armature constructions. It will be shown however that the "in air" solution above given is equally applicable to the case of Ruhmkorff coil constructions as well as to the analogous cases of core and shell type transformers and generally speaking to all cases where electrical windings are brought into proximity to iron masses.

For low as well as high frequencies it has become classic to assume that the self-induction flux of conductors placed near an iron surface enters the iron and on emerging penetrates the conductors parallel to such

iron surface. It has been for this reason that Sir J. J. Thomson's formula for copper strip (see Recent Researches in Electricity and Magnetism, page 281) has been invoked. In the case of slot wound conductors A. B. Field's formulas have been employed (see Proc. A. I. E. E., 24, 659, 1905). The method of assuming that at the bottom of the slot the magnetic intensity is zero occurs in all the literature (see for example E. Arnold, Die Gleichstrommaschine). As a matter of fact the flux within the conductor is never magnetically short circuited by the iron as the experiments referred to below will clearly indicate. It is for this reason that whenever current carrying conductors are placed near an iron surface of whatever type, it would be much more exact to say that the flux always circulates about the conductors as if they were free in air. Thus, *iron even in close contiguity to copper conductors never materially introduces self-induction flux-penetrations of the latter.* To prove this has been the object of the described experimentation herewith.

A slot three inches wide by four inches deep was cut in a sheet of galvanized iron. In the one case no flange was provided for the inner margin of the slot but in later experiments such flange was arranged without any material modifications in the character of the result obtained

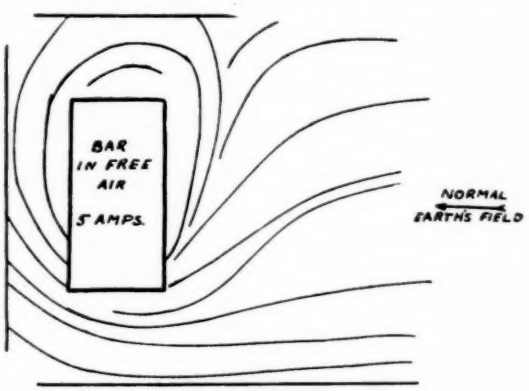

Fig. 1.

by the simpler arrangement. A bar of brass $2'' \times 1''$ as indicated in Fig. 2 was placed there in midway of the bar length of six inches so as to be one-half inch from the three walls of the slot. The plate and conductors were then arranged on a table and oriented so as to allow flux penetrations to occur relative to the three sides of the slot because of the presence of the earth's magnetic field. Such flux lines were explored and charted by means of a small watch charm compass of about five-sixteenths of an inch over-all diameter. The plate being horizontally arranged, the normal direction of the horizontal component of the earth's magnetic field is indicated by the arrow in Figs. 1 and 2. The thin lines

of Fig. 2 represent the flux distribution in the slot with no current flowing in the brass conductor. It will be observed that the flux lines of the earth's magnetic field penetrate the conductor in their path toward the sides of the slot. In other words there is a decided lateral component parallel to the bottom of the slot, but there is no evidence of lines passing from one side of the slot directly across to the opposite side of the slot which latter effect might have been presumed to exist if the compass needle was able to induce sufficiently strong poles of opposite polarity in the two parallel walls of the slot. The diagram therefore

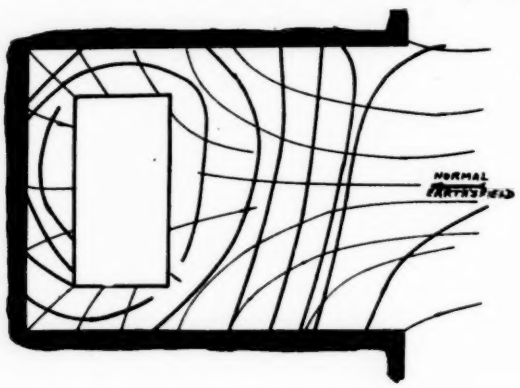

Fig. 2.

shows that with orientation settled upon the walls of the slot have substantially the same polarity due to the earth's magnetic field.

On passing a current of five amperes through the two square inches of brass the flux distribution in the slot radically changes. There is absolutely no evidence of lines of flux passing through the conductor from side to side to complete their circuit through the iron at the bottom of the slot. In fact a compass needle placed between the current carrying conductor and the sides of the slot is deflected so as to indicate that *the encircling lines actually pass up one side of the slot and down on the other.* Naturally, because of the presence of the earth's magnetic field a considerable skewing of flux lines occurs. The only place however where lines pass from side to side of the slot is above the conductor's surface. Between the conductor and the bottom of the slot lines do appear but they are parallel to the conductor and slot surfaces. As a control Fig. 1 was mapped out to show how the skewing of encircling flux lines occurs about the conductor due to the presence of the earth's magnetic field. In this case the same amount of current as before traversed the conductor.

Insofar as the effect of an alternating impressed E.M.F. upon a conductor is to crowd current toward the skin because of the self-induction

flux linkages within the conductor, it is evident that according to the established theory mentioned in the opening paragraphs of this paper that such skin effect on the basis of short circuiting of self-induction lines by the iron of the slot would necessarily crowd the current to the top of the conductor within the slot. However, if the self-induction lines within the conductor are practically unaffected by the presence of iron masses about the conductor, it will mean in the case of rectangular conductors that there will be four boundary surfaces where the skin effect will take place instead of one as in the previous instance.

As a further check on the above theory coupled with the experimentation an improved mathematical investigation was made along classical lines. In the work of A. B. Field (loc. cit.) and others the following two conditions were imposed.

1. The magnetic potential $\int H \cdot dl$ beneath the mth conductor counting from the bottom of the slot was due to $(m - 1)$ conductors carrying the normal load current. This meant first, that only a lateral component of self-induction flux existed, and secondly, that beneath the first conductor no magnetic field was set up.

2. That the aggregate current in the mth conductor was equal to the load current.

It will be seen that no limitation was set as to the magnetic potential at the top of the mth conductor. In the improved mathematical investigation conducted by the writer the second condition was replaced by the following:

2*a*. The magnetic potential at the top of the mth conductor was due to m conductor carrying the line current.

In this way no necessity arose for separately indicating the line current in the mth conductor.

A solution of the differential equation leads to the following value of the average resistance factor K_M for an M layer winding would therefore be

$$K_M = a \cdot l \left\{ \frac{\sinh 2al + \sin 2al}{\cosh 2al - \cos 2al} + \tfrac{1}{2}(M - 1) \frac{\sinh al - \sin al}{\cosh al + \cos al} \right\}.$$

It is to be observed that the above value coincides with Field's finding on the assumption of a single layer winding only.[1]

BERKELEY, CALIFORNIA,
November 13, 1919.

[1] It is desired to acknowledge the help of my assistant Mr. Shuichi Sumioka in conducting the above experiments.

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

MINUTES OF THE ST. LOUIS MEETING, DECEMBER 30, 31, 1919, AND JANUARY 1, 1920.

THE twenty-first annual meeting (the 101st regular meeting) of the American Physical Society was held at Soldan High School in St. Louis, Mo., on December 30, 31, 1919, and January 1, 1920, in affiliation with Section B-Physics—of the American Association for the Advancement of Science. The local arrangements were such that the social features were especially enjoyable, a pleasant occasion being the Physical Society Dinner at the Hotel Statler on the evening of December 30, attended by fifty-eight members and guests.

At the business session held on December 31, 1919, officers for 1920 were elected as follows: *President*, J. S. Ames; *Vice-President*, Theodore Lyman; *Secretary*, D. C. Miller; *Treasurer*, G. B. Pegram; *Managing Editor*, F. Bedell; *Councillors*, F. B. Jewett and Max Mason; *Members of the Editorial Board*, E. L. Nichols, C. W. Sparrow and W. F. G. Swann.

The question of the relation of the Society to the work of the Trustees for the Preparation of Critical Tables of Physical and Chemical Constants was brought before the Society; after discussion it was, by general consent, referred to the president, the councillor and the trustee representing the Society, for such action as may seem best.

At the meeting of the Council held on December 30, 1919, the following elections were made: *elected to regular membership*, T. H. Gronwall, E. H. Kennard, Henry A. McTaggart; *elected to associate membership*, William H. Agnew, W. H. Bair, Vola P. Barton, Henry M. Brock, J. T. Lindsey Brown, John A. David, E. C. Gaskill, Charles W. Henderson, F. F. Householder, Teizo Isshiki, Charles S. Jewell, P. Kirkpatrick, F. W. Kranz, Charles P. Miller, George S. Monk, Chalmer N. Patterson, Herbert J. Plagge, Geo. E. Raburn, S. P. Shackleton, George C. Southworth, John Alden Terrell, John A. Tobin, A. P. Vanselow, E. E. Zimmerman; *transferred from associate to regular membership*, Harold D. Babcock, Clifton G. Found, R. C. Gibbs, J. A. Gray, Frank B. Jewett, Edwin C. Kemble, Fred Loomis Mohler, Lindley Pyle, C. V. Raman, Paul E. Sabine, F. B. Silsbee, Elmer H. Williams,

On Tuesday afternoon, December 30, 1919, the president, J. S. Ames, delivered an address on "Einstein's Theory of Gravitation and Some of Its Consequences." This was a masterly presentation of the development and conclusions of this theory, and it was listened to by the largest audience of the meetings.

The session on the afternoon of Wednesday, December 31, 1919, was under the auspices of Section B-Physics of the American Association of the Advancement of Science. The retiring Chairman of Section B, Dr. G. F. Hull, gave an address on "Some Aspects of Physics in War and Peace." Following this there was a symposium of four special papers on "Phenomena in the Ultra-Violet Spectrum, Including X-Rays," by R. A. Millikan, D. L. Webster, Wm. Duane and A. W. Hull.

The programs consisted of thirty-four papers, six of which were read by title only, presented at four different sessions. The program of eight papers given at the session of Wednesday morning, consisted exclusively of papers relating to acoustics. The average attendance was about eighty-five, the maximum being about one hundred and twenty-five. The program was as follows:

Variation of Transparency to Total Radiation with Temperature of Source. S. LEROY BROWN.

The Dissipation of Heat by Various Surfaces in Still Air. T. S. TAYLOR.

The Influence of Air Velocity and the Angle of Incidence on the Dissipation of Heat. T. S. TAYLOR.

The Measurement of Thermal Expansion of Metals at Ordinary Temperatures. CHARLES D. HODGMAN.

A Method for Determining the Photographic Absorption of Lenses. G. W. MOFFITT.

Defects in Centered Quadric Lenses. IRWIN ROMAN.

The Sinker Method Applied to the Rapid and Accurate Determination of Specific Gravities. N. W. CUMMINGS. (Read by title.)

Amplification of Currents in the Bunsen Flame. C. W. HEAPS.

A new Type of Non-Inductive Resistance. H. L. DODGE.

Some Laboratory Uses for the Contact Rectifier. J. C. JENSEN.

An Undamped Wave Method of Determining Dielectric Constants of Liquids. W. H. HYSLOP, and A. P. CARMAN. (Read by title.)

Difficulties in the Theory of Rain Formation. W. J. HUMPHREYS.

A Physical Theory of Ocean or Reservoir Temperature Distributions, Regarded as Effects of Solar Radiation, Evaporation, and the Resulting Convection. GEO. F. MCEWEN.

Electromagnetic Induction and Relative Motion. W. F. G. SWANN.

The Influence of Blowing Pressure on Pitch of Organ Pipes. ARTHUR C. LUNN.

A Photographic Study of Explosions in Gases. JOHN B. DUTCHER.

A Photographic Study of Sound Pulses through Crooked and Curved Tubes with Deductions Concerning Telephone Mouthpieces, Phonograph Horns, etc. ARTHUR L. FOLEY.

A Photographic Method of Measuring the Instaneous Velocity of Sound Waves at Points near the Source. ARTHUR L. FOLEY.

A Possible Standard of Sound—I, Study of Operating Conditions; II, Study of Wave Form. CHAS. T. KNIPP.

The Performance of Conical Horns. G. W. STEWART.

A Photographic Study of the Wave-Form of Sounds from Large Guns in Action. DAYTON C. MILLER.

The Calibration of a Sound Chamber and Sound Sources and the Measurement of Sound Transmission of Simple Partitions. PAUL E. SABINE.

Transmissions of Sound Through Walls. F. R. WATSON.

Charcoal Absorption and Cyclic Changes. THOS. E. DOUBT.

The Heat of Vaporization and Work of Ionization. C. S. FAZEL. (Read by title.)

Energy Content of Characteristic Radiations. CHESTER W. RICE.

THE Spectrum of Radium Emanation. R. E. NYSWANDER, S. C. LIND and R. B. MOORE.

The Zeeman Effect for Electric Furnace Spectra. ARTHUR S. KING. (Read by title.)

Critical Potentials of the "L" Series of Platinum. DAVID L. WEBSTER.

On the Possibility of Pulling Electrons from Metals by Powerful Electric Fields. R. A. MILLIKAN and B. E. SHACKELFORD.

On the Recoil of Alpha Particles from Light Atoms. L. B. LOEB. (Read by title.)

Reactive Hydrogen in the Electrical Discharge. GERALD L. WENDT and ROBERT S. LANDAUER. (Read by title.)

The Construction and Calibration of a Device Permitting the Application of a Current Pulse for a Predetermined Number of Milliseconds. LYNDLEY PYLE.

The Spectral Transmission of Various Glasses. HENRY P. GAGE.

DAYTON C. MILLER,
Secretary.